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Concept Evaluation: Field Feedback Computation of Electromagnetic Scattering

by

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translates the body-surface currents into the scattered near-field. The thin-wire scattering problem is formulated in terms of this field-feedback method. Results of this computation are compared in graphical form to those found by Hallen's integral equation solution of the scattering problem for several cases of interest.

Concept Evaluation: Field Feedback Computation of Electromagnetic Scattering

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ABSTRACT

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I. INTRODUCTION

The objective of this endeavor is to investigate the performance of a new, and potentially quite powerful, technique for the computation of electromagnetic scattering problems. For reasons that will later become apparent, this new technique will be denoted the "field feedback formulation" (FFF or F^3). The work reported here is a first application of F^3, where the calculation of scattering from straight thin-wires has been performed. This is an almost classical electromagnetic field problem that has been routinely solved by use of integral equations for many years. Such a cononical problem was chosen as a testing vehicle for the newly conceived F^3 analysis method since alternate integral equation solutions are readily available for comparisons.

The formulation developed here employs a combination of the finite element numerical method and the classical magnetic vector potential. The finite element method (FEM) is an application of variational techniques which implements a weighted residual approximation. The high accuracy of the finite element method is used to advantage in reducing the amount of computing required, as compared to other numerical methods. Use is also made of coupled azimuthal potentials (CAP). The CAP formulation restricts the class of approachable

problems to those which are axially symmetric, but does allow the treatment of material inhomogeneities and multiwavelength dimensions.

Thin-wire assumptions are made which greatly simplify the current-scattering problem. These assumptions, however, are by no means critical to the solution method. The formulation should be readily extendable to a large class of important interior, radiation, and scattering problems.

II. THE PROBLEM AND METHOD OF SOLUTION

Electromagnetic scattering problems usually involve a known incident field, such as a uniform plane wave. The scattered field can be found from the current induced by the presence of the incident field, but the induced current can be found only after the total field is known. Since the total field is just the sum of the incident and scattered fields, a definite difficulty has been encountered. Knowledge of the induced current is needed to find the scattered field, while at the same time, the scattered field must be known to determine the induced current.

This is clearly seen in the form of Maxwell's equations for the scattering problem. Assume that a scatterer with properties $Y_1=\sigma_1+j\omega\varepsilon_1$, $Z_1=j\omega\mu$, is in free space where $Y_o=j\omega\mu_o$, $Z_o=j\omega\mu_o$, and in the presence of sources $\overline{J}_o$, $\overline{M}_o$. Then the curl equations have the form

$$\nabla x\overline{H} = \overline{J}_o+Y_1\overline{E} \tag{1a}$$

$$\nabla x\overline{E} = -\overline{M}_o-A_1\overline{H} \tag{1b}$$

where the induced electric and magnetic currents are

$$\overline{J}_1 = (Y_1-Y_o)\overline{E} \tag{2a}$$

$$\overline{M}_1 = (Z_1-Z_o)\overline{H}$$

and the scattered magnetic field is

$$\bar{H}_{scat} = \bar{H} - \bar{H}_{incident} \tag{3}$$

The electric field relationship is, of course, similar.

The simple and compact nature of the simultaneous equations (1) and (2) is highly appealing, but of little practical use in obtaining numerical answers to a problem because of the excessive computing required. Yet, while the requirement to have two unknowns cannot be escaped, the equations relating these unknowns can at least be rendered more tractable.

A. THE FIELD FEEDBACK FORMULATION

As will be shown later, two simple matrix equations can be derived which relate the magnetic and electric fields at a number of points along the surface of the scatterer to the fields at points along a boundary some distance away. These equations are independent and work in opposite directions. That is, from the first equation, given the total boundary field values, $\underline{e}_t$ and $\underline{h}_t$, the scatterer surface total field values, $\underline{e}_\omega$ and $\underline{h}_\omega$, can be found:

$$[\underline{e}_\omega \,\vert\, \underline{h}_\omega]^T = \underset{\sim}{V} \, [\underline{e}_t \,\vert\, \underline{h}_t]^T \tag{4a}$$

The column vectors, $[\underline{e}_i \,\vert\, \underline{h}_i]^T$, contain the field values at the computation points. The second equation takes $[\underline{e}_\omega \,\vert\, \underline{h}_\omega]^T$ and yields the scattered field values $[\underline{e}_s \,\vert\, \underline{h}_s]^T$ along the boundary:

$$[e_s \,|\, h_s]^T = \underset{\sim}{T}\, [\underline{e}_\omega \,|\, \underline{h}_\omega]^T \tag{4b}$$

Equations (4a), (4b), and (3) comprise a simple feedback system, as in Figure 1.

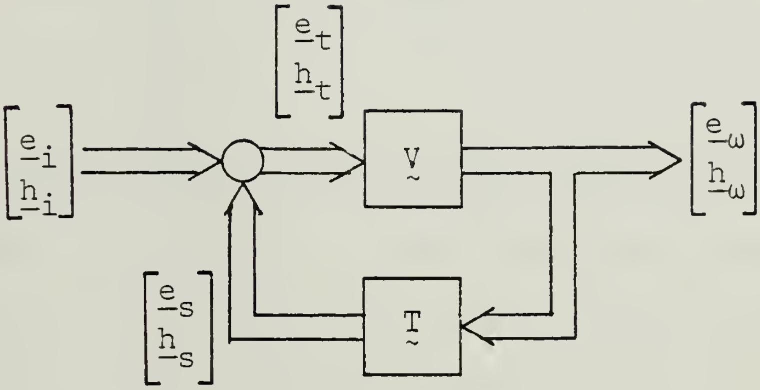


Figure 1. Field Feedback

What remains, then, is to construct the matrices $\underset{\sim}{V}$ and $\underset{\sim}{T}$ to fit the problem at hand. It should be pointed out that the magnetic field alone is involved in the wire-scatterer problem as developed here. So $[\underline{e}_x \,|\, \underline{h}_x]^T$ is replaced by $\underline{h}_x$. The field translation matrices $\underset{\sim}{V}$ and $\underset{\sim}{T}$, though developed in exactly the same way, will be considerably simplified as compared to the general scatterer case.

B. THE WIRE SCATTERING PROBLEM

Given a perfectly conducting thin wire in free space, and a plane wave of known amplitude incident from a known direction, it is required to find the current induced on the wire and the

field scattered by the wire at any point in space. Finding the induced current is equivalent to finding the field on the wire surface, since $\bar{J} = \hat{n} \times \bar{H}$, where $\hat{n}$ is the unit vector normal to the surface. Normalized coordinates will be used throughout the problem, and are given by

$$(R,Z,\phi) = (K_o r,\ K_o z,\ \phi).$$

The coordinates (r,Z,ϕ) are standard circular cylindrical coordinates, and $K_o = 2\pi/\lambda_o$ is the free space wavenumber of the time-harmonic field. The situation is depicted in Figure 2. Use of these normalized coordinates will allow the formation of a compact and useful form of the CAP equations.

The wire radius is $\underset{\sim}{a}$, given in terms of λ_o, the free space wavelength. Then the normalized radius is $R_a = 2\pi\underset{\sim}{a}/\lambda_o$. The wire length is L, also given in wavelengths. So $Z_L = 2\pi L/\lambda_o$. The incidence angle of the plane wave is α, measured as shown in Figure 2. Since perfect conductivity is assumed, fields and currents will exist only on the surface of the wire. And since a very thin wire will be assumed, where $\underset{\sim}{a} < \lambda_o/10$, the current at the wire ends may be set to zero.

The problem will not be solved in terms of the usual electric and magnetic fields, $\bar{E}$ and $\bar{H}$, or the scalar potential, V. Instead, use will be made of the magnetic vector potential, $\bar{A}$. The current and scattered fields will be presented and calculated in terms of the CAP formulation discussed below.

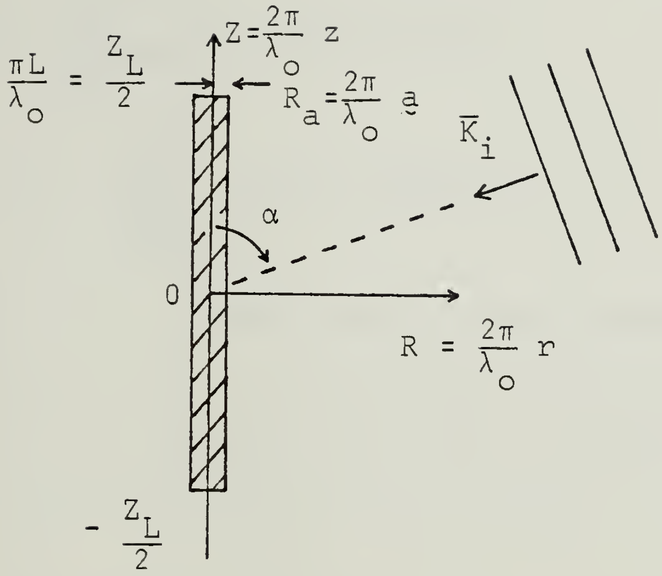


Figure 2. Normalized Coordinates

C. COUPLED AZIMUTHAL POTENTIALS

The CAP formulation [3] has been developed to handle previously intractable time-harmonic field problems involving continuously and discretely inhomogeneous bodies of revolution. With this limitation of axial symmetry, all electromagnetic field components can be expanded via exponential Fourier series in the ϕ coordinate. The model fields can then be represented by two coupled azimuthal potentials. These potentials satisfy a system of coupled partial differential equations, as well as a variational criterion, which will be developed here.

Using the normalized coordinates in Figure 2, the total fields are decomposed into azimuthal modes:

$$\bar{E}(R,Z,\phi) = \sum_{m=-\infty}^{\infty} \bar{e}_m(R,Z)\exp(jm\phi) \tag{5a}$$

$$\eta_o\bar{H}(R,Z,\phi) = \sum_{m=-\infty}^{\infty} \bar{h}_m(R,Z)\exp(jm\phi) \tag{5b}$$

where $\eta_o = 120\pi$. By substitution of these expansions into Maxwell's equations, it can be shown that all modal field components can be generated from two coupled azimuthal potentials $\psi_1(R,Z,m)$ and $\psi_2(R,Z,m)$, using

$$\hat{\phi} \times \bar{e}_m = jf_m(m\hat{\phi}\times\nabla\psi_1 - R\mu_r\nabla\psi_2) \tag{6a}$$

$$\hat{\phi} \cdot \bar{e}_m = \psi_1/R \tag{6b}$$

$$\hat{\phi} \times \bar{h}_m = jf_m(m\hat{\phi}\times\nabla\psi_2 + R\varepsilon_r\nabla\psi_1) \tag{6c}$$

$$\hat{\phi} \cdot \bar{h}_m = \psi_2/R \tag{6d}$$

where $\hat{\phi}$ is the unit azimuthal vector, ∇ is the two dimensional gradient operator, and

$$f_m[R,Z] = [\mu_r(R,Z)\varepsilon_r(R,Z)R^2 - m^2]^{-1} \tag{7}$$

The Euler-Lagrange variational technique will be discussed more fully in the exposition of the finite element method below. It is sufficient to note here that this technique finds the solution of the system partial differential

equations as the stationary point of a functional of the potentials and their first derivatives. This stationary point is defined as the point (ψ_{10}, ψ_{20}) at which the variation of F, δF, vanishes for perturbations $(\delta\psi_1, \delta\psi_2)$ about that point.

The functional F is a surface integral of the Lagrangian L, where

$$L = f_m[\nabla\psi_1\cdot(R\varepsilon_r\nabla\psi_1+m\hat{\phi}x\nabla\psi_2)+\nabla\psi_2\cdot(R\mu_r\nabla\psi_2-m\hat{\phi}x\nabla\psi_1)]$$

$$- (\varepsilon_r\psi_1{}^2+\mu_r\psi_2{}^2)/R \tag{8}$$

The functional F is integrated over the constant azimuth planar meridian cross section Ω of the volume in which the solution is of interest.

It is convenient at this point to change variables, and introduce a simplifying assumption. From (6b) and (6d) we have $e_{\phi m}(R,Z)=\psi_1/R$ and $h_{\phi m}(R,Z) = \psi_2/R$. So finding only the azimuthal components of the modal fields, $e_{\phi m}$ and $h_{\phi m}$, will yield a complete solution through (6a) and (6c). Now, since a thin wire is assumed, the phase variation of the incident field from side to side on the wire may be neglected. Then the induced current will be a function of Z alone; $\bar{J}=J_Z\hat{Z}$. Therefore the scattered fields will be azimuthally invariant, so that $\bar{E}(R,Z,\phi) = \bar{e}_o(R,Z)$ and $\eta_o\bar{H}(R,Z,\phi)=\bar{h}_o(R,Z)$. Furthermore, since $J_Z=\hat{n}x\bar{H}=H_\phi Z$ and since

$$\bar{E} = \frac{1}{j\omega\varepsilon} \nabla \times \bar{H} = \frac{1}{j\omega\varepsilon} [-\hat{R} \frac{\partial H_\phi}{\partial Z} + \hat{Z} \frac{1}{R} \frac{\partial}{\partial R} R H_\phi],$$

it follows that $e_{\phi o} = 0$, and $\bar{h}_o = h_{\phi o}$. The number of modes has now been reduced to one, the $h_{\phi o}$ mode.

Observe that (6a) and (6c) no longer involve coupled potentials. This allows the Lagrangian to be decoupled, and it may now be written

$$L = f_o\{\nabla(Rh_{\phi o})\cdot[R_{\mu r}\nabla(Rh_{\phi o})]\} - \mu_r Rh_{\phi o}^2. \tag{9}$$

So the functional is

$$F = \int_\Omega L(R,Z,h_{\phi o},\nabla h_{\phi o})dRdZ \tag{10}$$

It remains to find $h_{\phi o}(R,Z)$ to make F stationary in Ω for given Dirichlet boundary conditions. This will be done by use of the finite element method.

III. COMPUTATIONAL PROCEDURES

The constructions of the $\underset{\sim}{V}$ and $\underset{\sim}{T}$ matrices have little in common other than the coordinate system. The finite element method is used to generate $\underset{\sim}{V}$. $\underset{\sim}{T}$ is derived from the magnetic vector potential in this perfect conductor case, while if a dielectric body were used, both electric and magnetic vector potentials would be required to construct $\underset{\sim}{T}$. The use of numerical methods obviously implies that the current and fields will be found only at discrete points. As discussed in Section II.C., only points in a constant azimuth meridian plane need be considered. The region between the wire surface and the boundary at which the fields are to be calculated is divided into triangular elements. These are areas of integration in the finite element method. The divisions and the nodal numbering scheme are shown in Figure 3. There are a total of N_W points along the wire, and N_B points along the boundary.

A. FINITE ELEMENT METHOD

This method is one whereby the region of interest is divided into a number of overlapping subregions of a specified shape. Then the problem solution function θ is approximated by defining the solution as a linear combination of the products of unknown coefficients with "basis" functions defined over

over each of the overlapping subregions. These basis functions are usually sectionally linear, and, in very simple terms, are used to "fill in" the approximation between points. A weighted residual approximation of the solution θ, commonly known as the Ritz method, can then be made by the application of the discretized Euler-Lagrange variational procedure, subject to the appropriate boundary conditions. The rigorous development of each of these parts of the finite element method is not necessary here, but the background required for a basic understanding of the method will be given below.

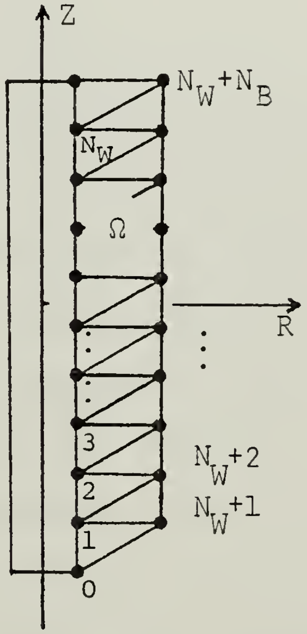


Figure 3. Solution Points

1. Basis Function Expansions

In order to solve a problem numerically, some method of approximating the solution using a finite number of unknowns must be found. Also, the method's convenience must be balanced against the desired accuracy. A general term for such an approximation is a "basis function expansion" of a function θ [2]. The type of expansion used in the finite element method for a one-dimensional problem is given by

$$\theta_N(X) = \sum_{i=1}^{N} \theta_i u_i(X) \qquad (11)$$

where θ_N is the discrete version of θ, the θ_i are the values of θ(X) at the discrete points X_i, and the $U_i(X)$ are the basis functions at the points X_i. Linear basis functions are usually employed in FEM calculations, though quadratic or higher-order polynomial functions could be used to obtain greater accuracy. Linear basis functions for a one-dimensional problem are shown in Figure 4. Note that the node spacing is irregular. This allows tailoring of the expansion to better approximate the solution, and, in two or higher dimensional cases, may be necessary because of the shape of the solution region. If neither of these are necessary, regular node spacing can be used with some advantage to reduce complexity. The basis fucntions u_i are seen

to have unit value at their respective nodes x_i, and are zero for $x_{i-1} \geq x \geq x_{i+1}$, $1 \leq i \leq N$.

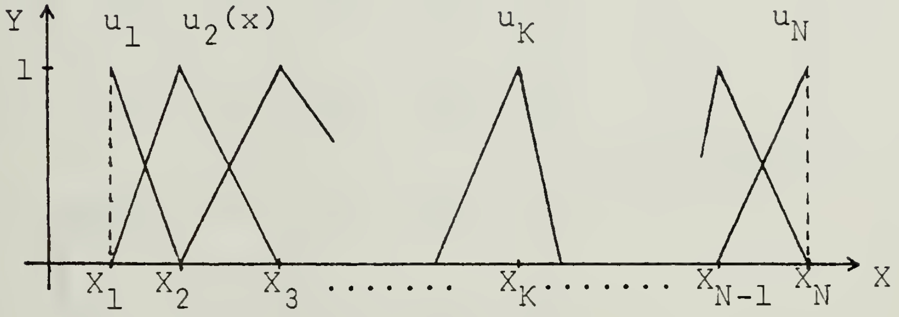


Figure 4. Linear Basis Functions

A two-dimensional linear basis function often used is the "pyramid" function; its two-dimensional projection is shown in Figure 5. A rectangular region Ω in the x-y plane has been discretized by division into the triangular elements shown. The boundary $\partial\Omega_{32}$ in Ω of the function $u_{3,2}(x,y)$ has been outlined to make the shape clear.

The function has a value of one at the center node, and is zero along and outside the periphery. Its equation can be written

$$u_{32}(x,y) = u^j_{32}\ (x,y),\ j = a,f \qquad (12)$$

where the component functions are planar segments of the form

$$u^j_{32} = \alpha x+\beta y+\gamma \qquad (13)$$

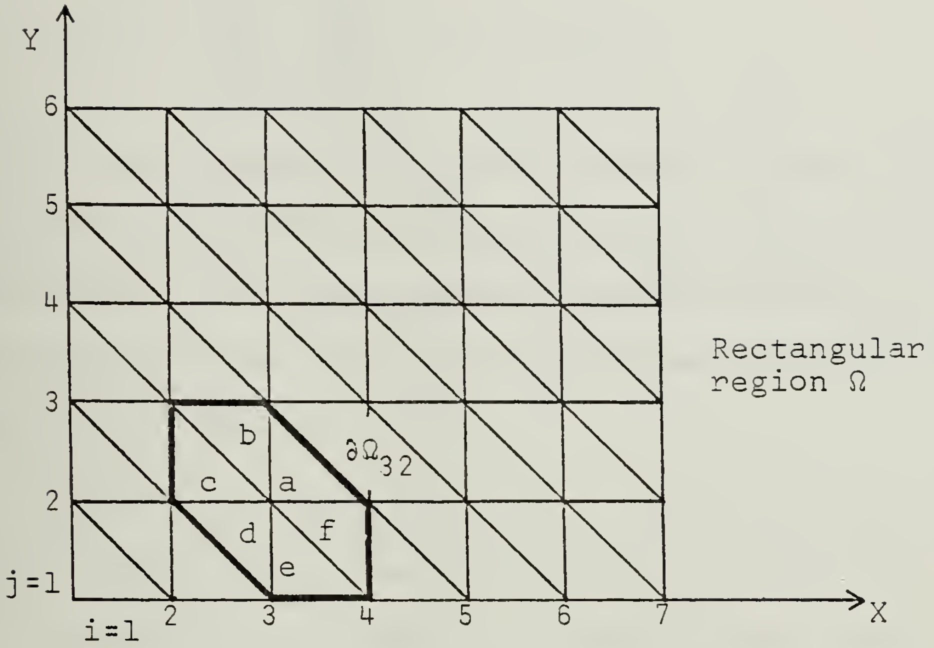


Figure 5. Two-Dimensional Finite Element Mesh

Then α,β,γ must be functions of (x,y) so that $u^j_{32}(x_3,y_2)=1$, $u^j_{32}(x_i,y_j)=0$, $i\neq 3$, $j\neq 2$, for integer (i,j).

As in the one-dimensional case, the basis functions overlap; that is, a function is centered around each node in the region or on the boundary. Of course, on the boundary of Ω at least half of the function is "lost" outside Ω. The overall appearance of the basis function array in three dimensions might be imagined as a crystalline surface of multifaceted peaks centered over the nodes, with raised valleys running between. If a function θ(x,y) were defined over Ω,

its expansion could be written

$$\theta_N(x,y) = \sum_{i=j}^{7} \sum_{j=1}^{6} \theta_{ij} u_{ij}(x,y) \tag{14}$$

If the exact values of θ_{ij} were known, θ_N would be a multi-faceted approximation to $\theta_{(x,y)}$, equal only at the nodes x_{ij}. Then, within the limitations of machine accuracy and computing time, the greater the node density the better the approximation θ_N will be.

The wire scattering problem expands the field, $h_{\phi o}$, by use of these pyramid basis functions. Referring to Figure 3, it is seen that the region Ω covered by the triangular elements will not contain any complete pyramids, but this is not really necessary. It will prove convenient to use a sequential node coordinate system rather than the (i,j) Cartesian description of (12). Let n be the node coordinate, where $1 \leq n \leq N$, $N=N_W+N_B$. Then the approximating expansion $h_{\phi N}$ of $h_{\phi o}$ can be written

$$h_{\phi N}(R,Z) = \int_{n=1}^{N} h_n u_n(R,Z) \tag{15}$$

Note that the top and bottom wire nodes are not represented. This is because of the assumption that $h_{\phi o}=0$ at these points.

Two field vectors can now be defined. The nodal wire-surface field is

$$\underline{h}_w = \underline{h}_n \quad , \quad n = 1, N_w \tag{16}$$

The nodal boundary field is

$$\underline{h}_B = \underline{h}_n \quad , \quad n = H_w+1, N \tag{17}$$

These vectors contain the approximate values of $h_{\phi o}$ at the nodes of the triangular mesh.

2. Euler-Lagrange Variational Technique

Consider the differential equation $A\theta=f$, which together with certain boundary conditions, describes some physical problem in a region Ω. The complex scalar wave (Helmholz) equation, $A = [\nabla^2+k^2]\theta=0$ is an example. The differential equation is related to the functional $F(\theta)$, where

$$F(\theta) = \int_\Omega L(\theta, D_x\theta, D_y\theta, x, y)d\Omega \tag{18}$$

This relationship is a fundamental result of the calculus of variations, and simply stated, is the fact that the functional $F(\theta)$ is stationary at the "point" θ_o which also satisfies the differential equation. That is, if the particular function θ_o can be found which causes the derivative (or first variation) of $F(\theta)$ to vanish, then $F(\theta)$ is said to be stationary, and θ_o is said to be the stationary point of $F(\theta)$; θ_o is then the solution of the equation $A\theta=f$ [5].

Observe that the vanishing of $dF/d\theta$ implies that a minimum, maximum, or saddle point of F occurs $\theta=\theta_o$. In many physical problems F is associated with the potential energy of systems and thus finding θ_o is sometimes termed the

"minimization" of F. The problem now presents a choice of computational methods. The differential equation might be approximated by a discrete system, say finite differences, or the variational integral, F, can be "minimized" over a finite number of trial functions, as is done by the finite element method.

It is necessary in this procedure to find the Lagrangian L for use in F(θ) [4]. This L must be such that its insertion in the Euler-Lagrange equation produces the differential equation that is to be solved. As an example, the Euler-Lagrange equation for the two independent variable case is given by

$$\frac{\partial L}{\partial \theta} - \frac{\partial}{\partial X}\left[\frac{\partial L}{\partial D_x \theta}\right] - \frac{\partial}{\partial Y}\left[\frac{\partial L}{\partial D_y \theta}\right] = 0 \qquad (19)$$

There is no definite way to discover L for any given problem, but it usually yields to informed guessing.

The problem's boundary conditions present no unmanageable difficulties in this formulation. If θ is specified on the boundary (Dirichlet boundary conditions) nothing else need be done. If θ is unspecified on the boundard, the stationary solution will automatically fulfill the Neumann boundary condition, $\frac{d\theta}{d\hat{n}} = 0$, also known as the natural boundary condition. Boundary conditions of a more complicated type will require the functional F to be modified, but this procedure will not

be necessary here. It can be easily shown by using the field generating equations in [6], that the natural boundary condition here results in a zero tangential electric field on the wire surface.

3. Application to the Thin-Wire Problem

As discussed in Section II, the driving force in the scattering problem is the incident plane wave. Then, the boundary conditions to be applied are the natural condition along the wire, for nodes n=1, N_w, and a Dirichlet condition along the outer boundary of Ω; $h_{\phi o}=h_t$ (total field). The Dirichlet condition $h_{\phi o}=0$ has also been imposed on the top and bottom wire nodes. These correspond to boundary conditions along the top and bottom node rows of a larger mesh, as in Figure 5.

The minimization of F for the wire scattering problem is carried out as follows. F is discretized by substituting $h_{\phi N}$ for $h_{\phi o}$, which yields the approximate functional

$$F(h_{\phi N})=\int_{\Omega}[f_o\{\nabla(Rh_{\phi N})\cdot R\mu_r\nabla(Rh_{\phi N})\}-R\mu_r h_{\phi N}^2]dRdZ \qquad (20)$$

The exact functional $F(h_{\phi o})$ is made stationary about the solution $h_{\phi o}$ by differentiating F with respect to $h_{\phi o}$. This makes F stationary about every point on $h_{\phi o}$. Since the approximate solution is to be found only at a finite number (N_w) of points, the functional needs to be made stationary only at those points. This is done by a simultaneous solution of the N_w stationary condition equations

$$0 = \frac{dF}{dh_i}(h_{\phi N}) = 2\int_\Omega \left[\frac{1}{R}\nabla R\frac{dh_{\phi N}}{dh_i}\cdot\nabla Rh_{\phi N} - R\frac{dh_{\phi N}}{dh_i}h_{\phi N}\right]$$

$$dRdZ, \quad i = N_w \tag{21}$$

where h_i is the unknown value of $h_{\phi N}$ at the node n=i, and $\mu_r=1$. Writing $h_{\phi N}$ explicitly,

$$\int_\Omega \left[\frac{1}{R}\nabla Ru_i\cdot\nabla R\sum_{n=1}^{N} h_n u_n - Ru_i\sum_{n=1}^{N} h_n u_n\right]dRdZ=0, \quad i=1,Nw \tag{22}$$

or, since $\sum_{n=1}^{N} h_n$ is a linear operator with no coordinate dependence,

$$\sum_{n=1}^{N} h_n \int_\Omega \left[\frac{1}{R}\nabla Ru_i\cdot\nabla Ru_n - Ru_i u_n\right]dRdZ=0, \quad i=1,Nw \tag{23}$$

Boundary conditions at the top and bottom wire nodes are already included in these equations. Application of the total field boundary condition is accomplished by observing that the h_n, $n=N_w+1,N$ are known, where, using (15), $\underline{h}_B=\underline{h}_t$. So by splitting the sum the useful minimization equations for $F(h_{\phi N})$ are found to be

$$\sum_{n=1}^{N_w} C_{in}h_n + \sum_{n=N_w+1}^{N} C_{in}h_n = 0 \qquad , i=1,Nw \tag{24a}$$

where

$$C_{in} = \int_{\Omega} [\frac{1}{R}\nabla Ru_i \cdot \nabla Ru_n - Ru_i u_n] dRdZ \qquad (24b)$$

The matrix equation to be used for computation can be formed directly from (24). First, recall that the basis functions u_n are zero on and outside of the boundary $\partial\Omega_n$. This means that the product of non-overlapping basis functions is zero. For example, in Figure 3, $u_3 \cdot u_1 = 0$. Thus

$$u_i u_n = 0 \quad , \quad |n-i| \geq 2 \qquad (25a)$$

and

$$\nabla Ru_i \cdot \nabla Ru_n = 0 \quad , \quad |n-i| \geq 2 \qquad (25b)$$

This behavior is very useful in that it produces banded coefficient matrices. Special techniques can be used to invert large banded matrices very quickly. Writing out a few equations should make the matrix construction clear.

$$\begin{aligned}
i=1:&\quad C_{11}h_1 + C_{12}h_2 + C_{1N_w+1}\,h_{N_w+1} + C_{1N_w+2}h_{N_w+2} = 0\\
i=2:&\quad C_{21}h_1 + C_{22}h_2 + C_{23}h_3 + C_{2N_w+2}h_{N_w+2} + C_{2N_w+3}h_{N_w+3} = 0\\
i=3:&\quad C_{32}h_2 + C_{33}h_3 + C_{34}h_4 + C_{3N_w+3}h_{N_w+3} + C_{3N_w+4}h_{N_w+4} = 0
\end{aligned} \qquad (26)$$

The system equation can therefore be written

$$\underset{\sim}{A}\underline{h}_W = -\underset{\sim}{B}\underline{h}_B \qquad (27)$$

where the $N_W x N_W$ coefficient matrix $\underset{\sim}{A}$ contains the basis function integrals for $i=1,N_W, n=1,N_W$ and the $N_W x N_W+1$ matrix $\underset{\sim}{B}$ contains the integrals for $i=1,N_W, n=N_W+1,N$. Both $\underset{\sim}{A}$ and $\underset{\sim}{B}$ are diagonal and have the forms

$$\underline{A}: \begin{bmatrix} X & X & & & & \\ X & X & X & & 0 & \\ & X & X & X & & \\ & & & \ddots & & \\ & 0 & & X & X & X \\ & & & & X & X \end{bmatrix} \qquad \underline{B}: \begin{bmatrix} X & X & & & & & \\ & X & X & & 0 & & \\ & & X & X & & & \\ & & & & \ddots & & \\ & 0 & & & X & X & \\ & & & & & X & X \end{bmatrix}$$

The wire field $\underline{h}_W$ due to the boundary field $\underline{h}_B$ is now known, since

$$\underline{h}_W = -\underset{\sim}{A}^{-1}\underset{\sim}{B}\underline{h}_B = \underset{\sim}{V}\,\underline{h}_t \qquad (28)$$

Equation (28) is the forward path in the system of Figure 1. The actual calculation of the A_{in}, B_{in} will be discussed in Section III.C.

B. SCATTERED FIELD INTEGRAL

1. Vector Potential Formulation

The development of the feedback path is less complicated than the forward path construction given in III.A. The

thin-wire approximation is applied to the familiar classical vector potential $\overline{A}$, and a matrix equation formulation built on a one-dimensional basis function expansion. The result will be an approximation giving the near field produced by a wire-surface field or current.

From the inhomogeneous Helmholz equation

$$\nabla^2\overline{A} + k^2\overline{A} = -\overline{J} \tag{29}$$

it is seen, since $\overline{J}=J_Z\hat{Z}$ by assumption, that $\overline{A}=A_Z\hat{Z}$. Then it follows from the vector potential defining equation

$$\overline{H} = \nabla \times \overline{A} \tag{30}$$

that

$$H_\phi = -\frac{\partial}{\partial R} A_z(R,Z) \tag{31}$$

The azimuthal invariance of $\overline{H}$ has been made explicit in (31).

A commonly used expression for $\overline{A}$ [1] is given as an integration of a weighted point-source Green's function $G(\overline{r}',\overline{r})$ over the current source where

$$G(\overline{r}',\overline{r}) = \frac{1}{4\pi} \frac{e^{-jk_o|\overline{r}-\overline{r}'|}}{|\overline{r}-\overline{r}'|} \tag{32}$$

Then

$$\overline{A}(\overline{r}) = \int_{V'} \overline{J}(\overline{r}')G(\overline{r}',\overline{r})dV' \tag{33}$$

This general situation is shown in Figure 6.

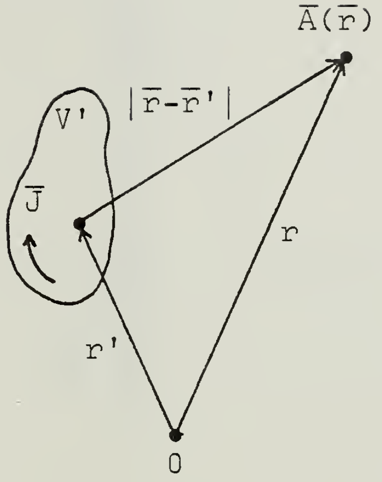


Figure 6

2. Scattered Field Matrix Equation

The formulation is easily applied to the thin-wire case. Recalling that $h_{\phi o} = \eta_o H_\phi$, and assuming that the current J_Z lies along the Z axis, $\overline{A}$, for the thin wire, is

$$\eta_o A_Z(R,Z) = \int\!\!\int_S h_\phi(Z') \frac{e^{-j|D|}}{4\pi|D|} R_a d\phi dZ' \qquad (34)$$

where, $|D| = \sqrt{R^2+(Z-Z)^2}$, as seen in Figure 7, and the integration is over S, the wire surface. Now, putting (34) in (31) yields

$$h_{\phi o}(R,Z) = \frac{R_a R}{2} \int_{-1/2\ Z_L}^{1/2 Z_L} h_{\phi o}(Z')\left[\frac{j}{|D|^2} + \frac{1}{|D|^3}\right] e^{-j|D|} dZ' \qquad (35)$$

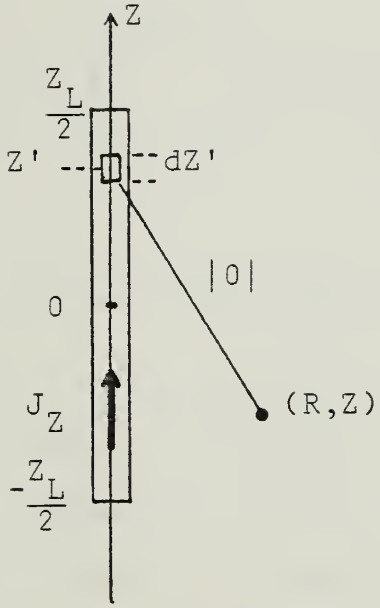


Figure 7

This $h_{\phi o}(R,Z)$ is approximately the scattered field, $h_s(R,Z)$, radiated by the wire carrying an induced current J_Z. This wire current, in the form of $h_{\phi o}(Z')$, must be approximated by a basis function expansion so that $h_s(R,Z)$ can be computed. Since $h_{\phi o}(Z')$ is a function of Z only, the one-dimensional linear basis functions shown in Figure 4 may be used. Then, for the same N_w nodes along the wire, as in Section III.A.,

$$h_{\phi o}(Z') = \sum_{n=1}^{N_w} h_n u_n(Z') \qquad (36)$$

Notice that (36) defines the same nodal surface-field vector as (16). Substitution of (36) into (35) produces

$$h_s(R,Z)=\frac{R_aR}{2}\sum_{n=1}^{N_W} h_n\int_{Z'_{n-1}}^{Z'_{n+1}} u_n(Z')\left[\frac{j}{|D|^2}+\frac{1}{|D|^3}\right]e^{-j|D|}dZ' \tag{37}$$

The limits of integration reduce from $\pm Z_{L/2}$ to Z'_{n-1}, Z'_{n+1} because the basis function $u_n(Z')$ is zero for $Z'_{n-1} \geq Z' \geq Z_{n+1}$. Equation (37) can be used to construct the $\underset{\sim}{T}$ matrix by noting that (37) gives the scattered field at one point (R,Z). Then, by choosing a number of points in some region, a discrete field approximation can be constructed. The points chosen will be, unsurprisingly, just those N_B points along the finite element mesh border. So the matrix equation is

$$\underline{h}_s = \underset{\sim}{T}\,\underline{h}_w \tag{38a}$$

where

$$T_{bn} = \frac{R_aR}{2}\int_{Z'_{n-1}}^{Z'_{n+1}} u_n(Z')\left[\frac{j}{|D|^2}+\frac{1}{|D|^3}\right] e^{-j|D|}dZ' \tag{38b}$$

is the coefficient of the n-th value of the wire field, and $T_{bn}h_n$ is one term of the b-th value of the scattered field.

Finally, since $\underline{h}_t = \underline{h}_s + \underline{h}_{inc}$, an expression for the total field in terms of the incident and wire fields is

$$\underline{h}_t = \underline{h}_{inc} + \underset{\sim}{T}\,\underline{h}_w \tag{39}$$

The scattered field $\underline{h}_s$ found in (38) is a near field expression. It is valid at any distance from the wire, but this behavior does not extend to the entire problem. If $\underline{h}_s$ were to be found at some large distance from the wire, simply extending the radial width of Ω would not suffice. This is because FEM calculation accuracy degrades as the distance from the wire to the boundary, B_L, increases. The magnitudes of the electrostatic and induction fields decrease rapidly as R increases, so the point is quickly reached where they become effectively zero as far as computer calculations are concerned. The resulting incorrect total field values are a part of the FEM R-related inaccuracies. The greater part of the FEM error is found in the basis function expansion of h_ϕ. The planar-section basis functions used are unable to accurately model the near field over a large radial distance, where "large", in this case, may be 1/100 of the wavelength.

To solve the far-field problem, the near-field solution must first be obtained for the wire field $\underline{h}_w$. Then (38) may be used to find $\underline{h}_s$ along any boundary. For typical far-field problems, assumptions may be made to reduce the integral's complexity. Let β be the angle of the far-field point, measured from the Z axis at the origin. Then $R=k_o r \sin\beta$, where r is the spherical radius of the point. The far-field assumption is

$$|D| = k_o r - Z' \cos\beta \quad \text{(in phase term of (38b))} \qquad (40a)$$

$$|D| = k_o r \quad \text{(in denominators)} \qquad (40b)$$

So the radiation field along a spherical arc of radius r is given by (38), where T_{bn} is

$$T_{bn} = j\frac{a\,\sin\beta}{2r}\,e^{-jk_o r}\int_{Z'_{n-1}}^{Z'_{n+1}} u_n(Z')e^{jZ'\cos\beta}\,dZ' \qquad (41)$$

C. PROBLEM IMPLEMENTATION

Combining equations (27) and (39) gives

$$\underline{h}_w = -(\underset{\sim}{A}+\underset{\sim}{B}\underset{\sim}{T})^{-1}\underset{\sim}{B}\;\underline{h}_{inc} \qquad (42)$$

This is, of course, the same equation as is represented in Figure 1, where $\underset{\sim}{V} = -\underset{\sim}{A}^{-1}\underset{\sim}{B}$. The system "transfer matrix" is defined by

$$\underline{h}_w = (\underset{\sim}{I} - \underset{\sim}{V} \cdot \underset{\sim}{T})^{-1} \cdot \underset{\sim}{V} \cdot \underline{h}_{inc} \qquad (43a)$$

Simple matrix algebra leads to a simplification of this result

$$\underline{h}_w = -(\underset{\sim}{A} + \underset{\sim}{B} \cdot \underset{\sim}{T})^{-1} \cdot \underset{\sim}{B} \cdot \underline{h}_{inc} \qquad (43b)$$

Multiplying this equation by $(\underset{\sim}{A}+\underset{\sim}{B}\underset{\sim}{T})$ yields (42), so the system equation as derived does behave as a feedback system.

The only variable remaining to be identified is $\underline{h}_{inc}$, the incident field vector. The incident plane wave h_{inc} can be

expressed in the normalized cylindrical coordinate system as

$$h_\phi(R,Z) = je^{-jZ\cos\alpha}\,J_o'(R\sin\alpha) \qquad (44)$$

where J_o' is the derivative of the zeroth-order Bessel function. Since R will be kept small, as discussed in Section III.B., (44) may be simplified by replacing $J_o'(R\sin\alpha)$ with the first few terms of its series expansion

$$J_o'(R\sin\alpha) \doteq -\left(\frac{R\sin\alpha}{2}\right)+\left(\frac{R\sin\alpha}{16}\right)^3-\left(\frac{R\sin\alpha}{384}\right)^5 \qquad (45)$$

Then $\underline{h}_{inc}$ may be calculated at the boundary nodes by use of (44) and (45).

Now that expressions for all of the quantities in (42) have been found, the computer program can be devised to carry out the calcuations. The use of an existing routine, CSMINV, for the inversion of $(\underset{\sim}{A}+\underset{\sim}{B}\underset{\sim}{T})$ reduces the number of major computations to two: equation (24b) for the $\underset{\sim}{A}$ and $\underset{\sim}{B}$ elements, and equation (38b) for the scattered field coefficients T_{bn}.

The integration of (34b) becomes algebraicly messy, but is otherwide straightforward. Because of the basis function's behavior, the integration area is reduced from Ω to $\partial\Omega$, the region of non-zero overlap of u_i and u_n. This integration over $\partial\Omega$ may then be written as a sum of integrations over the triangular elements comprising $\partial\Omega$, since u_n is piecewise planar in the elements. Also, note that C_{in} is independent of Z in

the sense that the integration for any pair (i,n) of nodes, will be the same if that pair is translated along the wire. This means that one wire node may be selected, and the integrals concerning that node and its immediate neighbors will be the only ones required to construct $\underset{\sim}{A}$ and $\underset{\sim}{B}$. For example, select i=2. Then C_{21}, C_{22}, C_{2N_w+2} and C_{2N_w+3} are the only integrals that need to be calculated. (Node three is not being neglected. Equation (24b) shows that $C_{23} = C_{32}$, but this is just C_{21} moved up one element. So $C_{23}=C_{21}$.)

In this simple narrow mesh case, the integration of (24b) is required over at most three elements (C_{22}), and the size and shape of all elements is the same. For many problems this will not be the case. For large meshes, and especially for skewed elements, it becomes very convenient computationally to change from a global to a local node system. Instead of moving from node to node, calculating all the coefficients about each node one at a time, an algorithm is devised to move from element to element, finding the integrals involving the three corner nodes over each element. This procedure effects a considerable simplification in the "bookkeeping" necessary to keep track of the nodal indices and coordinates.

The local node system employed here is shown in Figure 8. The integrals of the combinations of local nodes (k,l) are found over a upper and lower element. Then the integrals can be placed in their proper global positions. As an example,

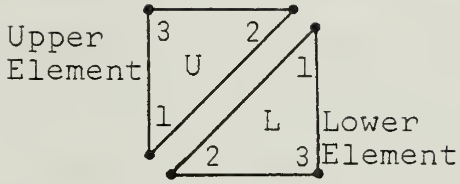


Figure 8. Local Node Indices

consider the upper element with global nodes $(2,3,N_W+3)$. The integrations of (24b) over U are found for local node pairs (1,1),(1,2),(1,3),(3,1),(3,2), and (3,3). Then the local node pair (1,1) integral, $\int(1,1)$ is put in $\underset{\sim}{A}$ as one part in three of C_{22}, $\int(1,2)$ is one part in two of C_{2N_W+3} in $\underset{\sim}{B}$, and $\int(1,3)=C_{23}$ in $\underset{\sim}{A}$. Also, $\int(3,1)=C_{32}$, $\int(3,2)$ is part of C_{3N_W+3}, and $\int(3,3)$ is part of C_{33}. The remaining three local node integrals over U are not needed in this case, but would be used in larger meshes.

For the thin mesh used here, the local node procedure improves only slightly the algorithm efficiency as compared to a global node procedure. The principle reason for its use was that an existing and successful subroutine was available to do local node integrations. This routine is named VARELA in the program. It requires only the local node ordering scheme and the coordinate (R,Z) of each local node to produce

a three by three matrix containing the integrations. As discussed above, the wire-scattering problem needs only two calls to VARELA to obtain the necessary values. The subroutine ABLOAD then fills $\underset{\sim}{A}$ and $\underset{\sim}{B}$ with these integrals.

The calculation of the coefficients T_{bm} is done by the familiar Gaussian quadrature method, using eight unequally spaced points. The interval of integration in (38b) is split into two segments, as in Figure 9. Then a change of variable is affected so that the limits of integration in each subinterval are X=+1 to -1. The Gaussian procedure then gives

$$\int_{-1}^{+1} f(x)dx \simeq \sum_{i=1}^{n} w_i[f(x_i)+f(-x_i)] \qquad (46)$$

where the values $\pm x_i$ and the weight factors w_i are tabulated in several references such as [6]. The code used to implement these integrations is contained in subroutine GFINT.

Equation (42) is coded in two subroutines, CFORM and ZCURR. Routine CFORM produces the matrix product $(\underset{\sim}{A}+\underset{\sim}{B}\underset{\sim}{T})^{-1}\underset{\sim}{B}=\underset{\sim}{C}$, while ZCURR calculates the incident field vector $\underline{h}_{inc}$ according to (44) and (45), then finds the wire field vector $\underset{\sim}{h}_w=\underset{\sim}{C}\,\underset{\sim}{h}_{inc}$.

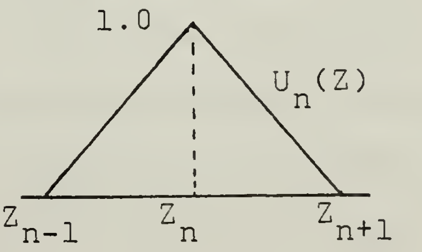


Figure 9

IV. RESULTS AND ANALYSIS

In many other time-harmonic field problems, FEM calculations yield good results with perhaps 20 to 30 nodes per wavelength. With higher node densities, say 50-75 perwavelength, the solution breaks down due to roundoff error in the computations. As will be shown, this does not appear to be the case with the field feedback formulation. Nodal densities as high as 100 per wavelength have been used without difficulty. Indeed, these high densities are required in order to achieve good convergence of the formulation. As yet, no satisfactory answer for this behavior has been found.

The only other parameter available to affect the convergence of the solution is B_L, the wire-to-boundary distance. As mentioned in section III.B.2., this parameter can be expected to be critical in achieving a good field approximation. Computed results show this to be true.

A great deal of effort was expended in looking for errors in the formulation before the solution behavior with respect to B_L and N_W was fully appreciated. It is still possible, no matter how unlikely, that some error still exists. If an error is present in the formulation, or code, it's effect is quite peculiar. By comparison with other FEM problems, the possible error causes two to three times the "normal" node

density to be needed, but otherwise seems to have no effect. That is, the results are satisfactory in all respects save node density required for convergence.

The primary comparison case chosen was a one-half wavelength wire with a radius of 1/50 wavelength. This case was a compromise forced by excessive required computing for longer wires and questionable integral equation comparison results with shorter wires. These other cases were looked at, and some example results are included here. Their investigation, however, has not been exhaustively completed due to lack of time.

A. INTEGRAL EQUATION COMPARISON

The basis used for judging the convergence of the FFM computation was a standard and successful type of integral equation formulation, Hallen's equation for scattering by a thin conductive cylinder. This method assumes a $\hat{Z}$ directed current density, and finds this wire current by a simultaneous solution of two expressions for the vector potential, $\overline{A}$. One equation is just equation (33). The second is a different form of equation (20)

$$\nabla^2\overline{A} + k^2\overline{A} = jw\varepsilon\, \overline{E}^s(\overline{r}) \qquad (47)$$

where $\overline{E}^s(\overline{r})$ is the scattered electric field. Simple manipulations yield the integral equation for I(z'), [7]

$$\int_0^L I(Z')G(Z,Z')dZ' - j\int_0^Z I(Z')\frac{Z(Z')}{Z_o} \text{ sink } (Z-Z')dZ'$$

$$+ C\sin kZ + D\cos kZ - U(Z) \qquad (48)$$

where A is an impedance per unit length, C and D are constants of integration, and U contains the incident field. Equation (48) is coded in subroutine INTEQ.

Results for the 1/2 x 1/50 wavelength case are shown on pages A-1 through A-38 of Appendix A. F^3 calculations are shown as a solid line, while integral equation computations are displayed as the "X" curve. The parameters of each case are given below the graph. The magnitude plots are scaled in milliamperes along x, and wavelengths along y. The wire is to be seen lying along the y axis, and is conveniently demarcated by the zero end values of the integral equation curve. The current phase plots have the same y axis units, and are scaled in degrees along x.

Graphs A-1 through A-24 show F^3 convergence with an increasing number of nodes. Values of N_W are 20,40,70 and 100, with three plane wave incidence angles (ALPHA) used in each case. In these plots, the plane wave is incident from the upper right for ALPHA=30 and 60 degrees. ALPHA=90° is broadside incidence. Convergence of the current magnitude

is plainly seen at all incidence angles. For $\alpha=90^\circ$, the error is 15.5% at the wire center for $N_w=20$, but only 2.5% for $N_w=70$. Similarly, the phase error decreases from 18° to 1°. Comparing the $N_w=70$ and $N_w=100$ cases, it is seen that the extra nodes make very little difference in the quality of the solution. Convergence has been essentially accomplished at $N_w=70$, and the additional memory and computing time for $N_w=100$ are not really necessary. The convergence is less satisfying for incidence angles other than 90°. At ALPHA=30°, for example, the magnitude is not skewed quite enough, while the phase error increases toward the wire bottom. This situation does not improve as N_w increases.

Graphs A-25 through A-38 show FFM convergence as a function of B_L. All F^3 calculations are made with $N_w=28$. The behavior shown in these plots is characteristic of all cases of wire length and radius. For B_L much too large, the magnitude is too low and the phase is off by as much as 90°. As B_L decreases, the magnitude rises, overshoots, and then settles back towards the correct, convergent value. The phase moves smoothly down to the correct value. Notice that the F^3 curves are nearly identical for DR=DZ/32 and DR=DZ/64. This implies that convergence has been reached by the time DR has decreased to DZ/32, or $B_L=.0005\lambda_o$. For this case the convergence point of B_L is actually about $B_L=.001\lambda_o$, as shown in A-1 through A-25. Results for several other cases of wire length and radius are shown in Appendix B. The one and two wavelength wire

calculations show that more nodes are required, assuming that $B_L=.001\lambda_o$ is still adequate since A_L has not changed. If 70 nodes are required for a good answer at $Z_L=\lambda_o/2$, then for two wavelengths as many as 300 nodes might be necessary. Similarly, well over 100 nodes could be needed for the one wavelength wire. The one wavelength case indicates that this is probably true. The F^3 solution for $N_w=100$ is perceptibly better than the $N_w=70$ solution, especially for off-axis incidence angles.

Appendix C contains graphs for two different cases: $A_L=1/100\lambda_o$ while $Z_L=\lambda_o/2$ or $Z_L=\lambda_o/4$. Graphs C-1 through C-4 again show convergence value is no longer $B_L=.001\lambda_o$, but is smaller than $B_L=.0005\lambda_o$.

Plots C-7 through C-10 show the quarter-wavelength wire case. It appears from the FFM curves that convergence has been almost reached for $N_w=40$. This calls into question the accuracy of the integral equation solution. This possible inaccuracy should not be too surprising, however, since the wire is relatively thick for it's length, and the integral equation does much better for very thin wires.

An "error isolation" check was made on the F^3 system by testing the feedback path. A fictitious sinusoidal wire current was generated and its radiated field calculated via $\underline{h}_s=\underset{\sim}{T}\,\underline{h}_w$. This field was compared to that given in reference [8]. The comparison is shown in plots C-11 and C-12. The two field calculations are equal, to the fourth significant digit. This indicates that the feedback path formulation and program coding is correct.

B. ERROR ANALYSIS

Consideration of the near field behavior reveals the reason for the critical dependence of the solution on B_L. The magnetic field magnitude in the very near field is dominated by a $1/R^2$ term, as can be deduced from equation (37). This is depicted in Figure 10. Also shown are two dashed lines which are the linear basis function expansion of the field. It is clear from the figure that the approximation of the field deteriorates as A_L decreases, if B_L is held constant. Thus it is seen why $B_L=.001\lambda_o$ is adequate for $A_L=.02\lambda_o$ while $B_L\approx.0005\lambda_o$ to obtain the best convergence with $A_L=.01\lambda_o$. It is noted, however, that even at this best convergence value of B_L, the F^3 solution is not as close to the integral equation solution as in the $A_L=.02\lambda_o$, $B_L=.001\lambda_o$ case. This fact can also be explained by consideration of the basis function approximation's behavior. While decreasing B_L improves the field approximation very near the wire, the approximation becomes worse in the neighborhood of $R=B_L$.

It thus appears that there is a limit to the usefulness of this formulation of the F^3 solution to the wire-scattering problem. The primary difficulty lies in the inaccuracy of the linear basis function expansion used in the FEM calculation. A much better basis function may contain a term proportional to $1/R^2$. This would greatly reduce the sensitivity of the solution to B_L.

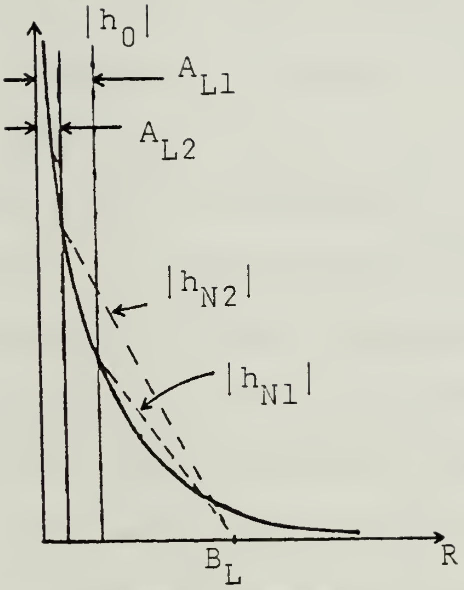


Figure 10

There is another source of error which is inherent in any sort of computer solution: roundoff. Simple calculations will be affected in the sixth or seventh significant digit by the use of single-precision arithmetic on the IBM-360. If only a few operations were required, this error could be ignored. But when solving large systems of equations many thousands of operations are required, and accuracy can be severely degraded.

A rough guide to the degree of trust to be placed in a solution can be had in the condition number of an inverted matrix. This number provides an upper bound on error propagation in a system due to the inaccuracy of finite-precision

arithmetic. For example, in the system $\underline{X} = \underset{\sim}{A}^{-1}\underline{Y}$, a condition number of 1000 for $\underset{\sim}{A}^{-1}$ would cause errors in the third significant digit of $\underline{Y}$ to appear as errors in the third, second or even first significant digit of $\underline{X}$.

This condition number is a measure of the similarity of the system equations. In the wire-scatterer case, as the solution points are placed closer together, the field values at these nodes become more indistinguishable. Hence the equations relating the field values at neighboring nodes become more nearly the same. The system solution, then, becomes sensitive to errors in the less significant digits.

Condition numbers of system matrices for the cases discussed here range from about 250 to 1000. Cases with fewer nodes and larger values of B_L have lower condition numbers, while as N_W increases and/or B_L decreases, the condition number increases rapidly. This is another reason why a good solution is more difficult to achieve for very thin or long wires. The use of a basis function more suited to the field behavior would allow larger values of B_L to be employed. This would lower the system condition number significantly, and improve the solution's accuracy and trustworthiness. Other than this and keeping N_W as small as possible, there is nothing that can be done to avoid the problems reflected in the condition number.

Even if the errors mentioned above could be eliminated, the results of both the F^3 and integral equation calculations

would still not represent the fields with complete accuracy. The assumptions made in both formulations are not strictly true for any wire of finite thickness. Current flows on the surface of the wire, not in the center, as assumed in the vector potential formulation.

Field phase variation from side to side of the wire is not zero, so there is an azimuthal component of current as well as the $\hat{Z}$-directed current assumed. And because this $\hat{\phi}$-directed current exists, the current is not zero at the wire ends. But, since essentially the same types of approximations are made for both solution methods, it is not possible to ascertain, from the comparisons made here, the degree to which either solution represents the actual fields.

V. CONCLUSIONS

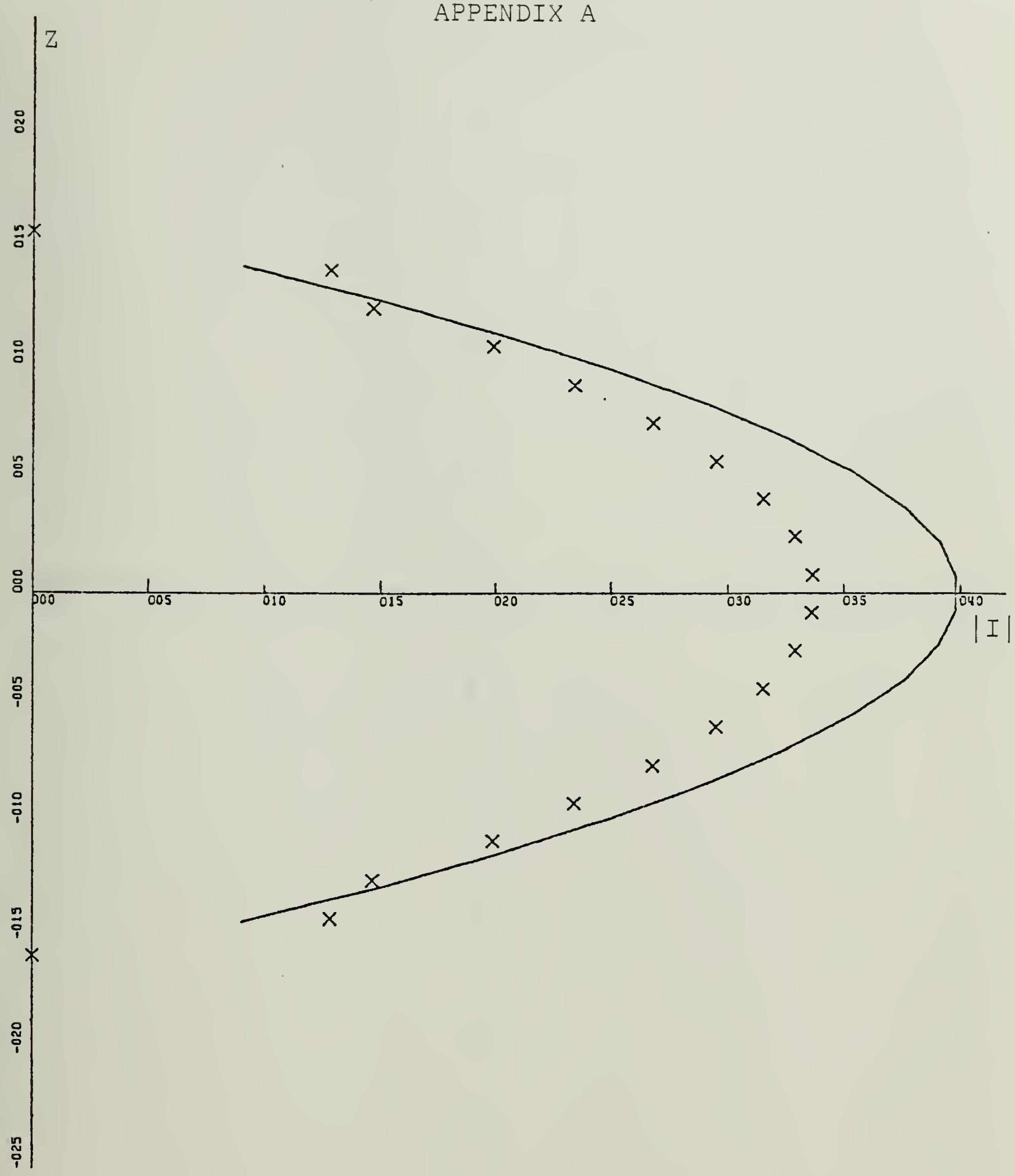
As a result of this investigation, the field-feedback formulation has been shown to be a sound technique for the computation of electromagnetic scattering problems. It has been demonstrated that F^3 calculations compare very favorably with results obtained by the conventional integral equation formulation. The success of this initial development is not unqualified, however. The two deficiencies previously discussed, basis function unsuitability and excessive number of required nodes, could present formidable difficulties in problems where a priori knowledge of field behavior is not available, or where bodies large compared to a wavelength are used.

The discrepancy between required and expected node density is particularly troublesome, as this fact is as yet unexplained. If high node densities are actually required by the method and are not just the result of some formulational error, then the attendant large matrices will limit the usefulness of F^3 computations because of the great amount of computer memory required.

In any case, the flexibility of the finite-element method when used in the coupled-azimuthal potential formulation should allow numerous heretofore difficult problems to be attacked. The ease with which mixed boundary conditions can

be met for complicated inhomogeneous scattering bodies, and the accuracy and simplicity of the vector potential formulation promise a wide applicability for the field feedback formulation.

APPENDIX A



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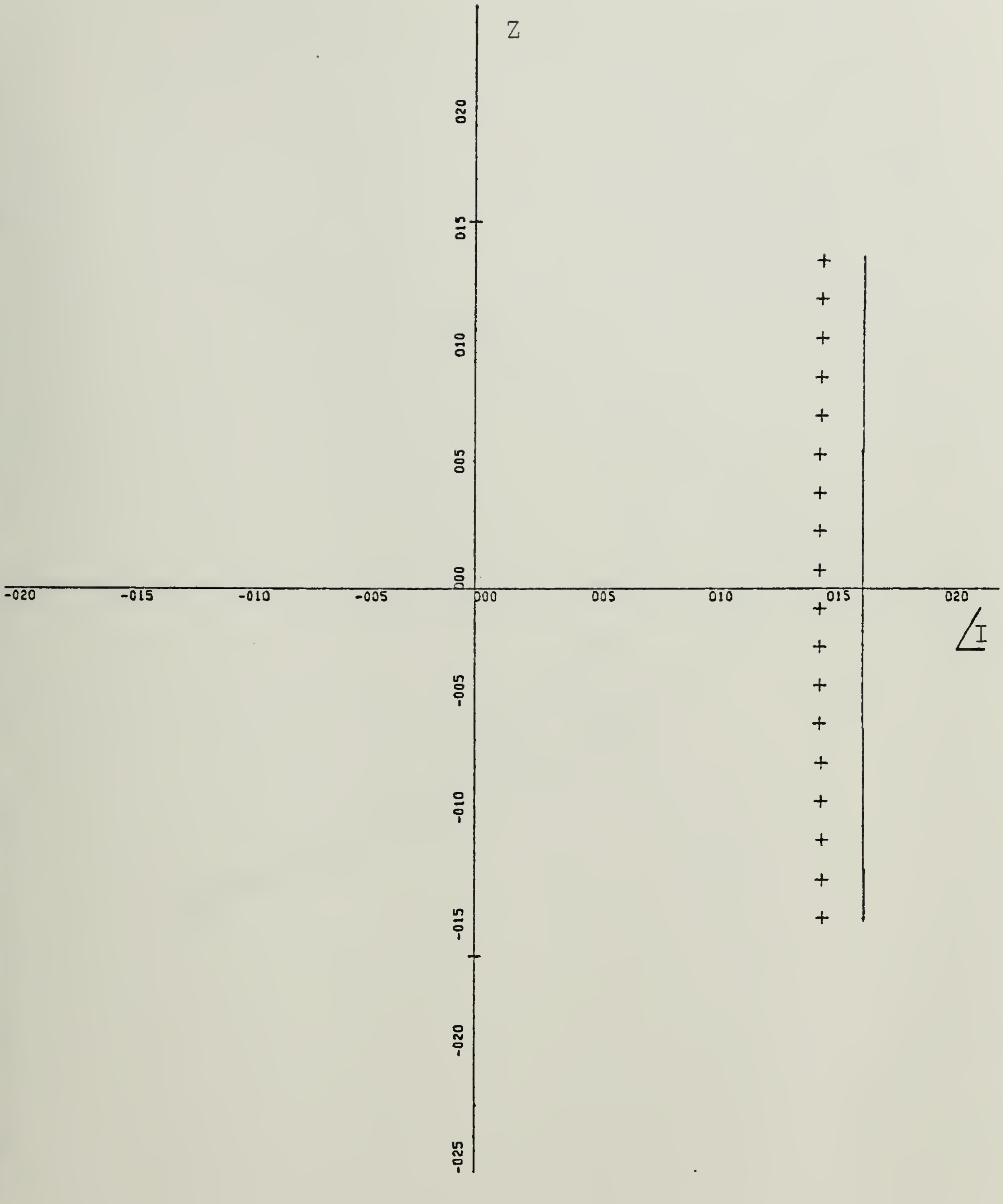

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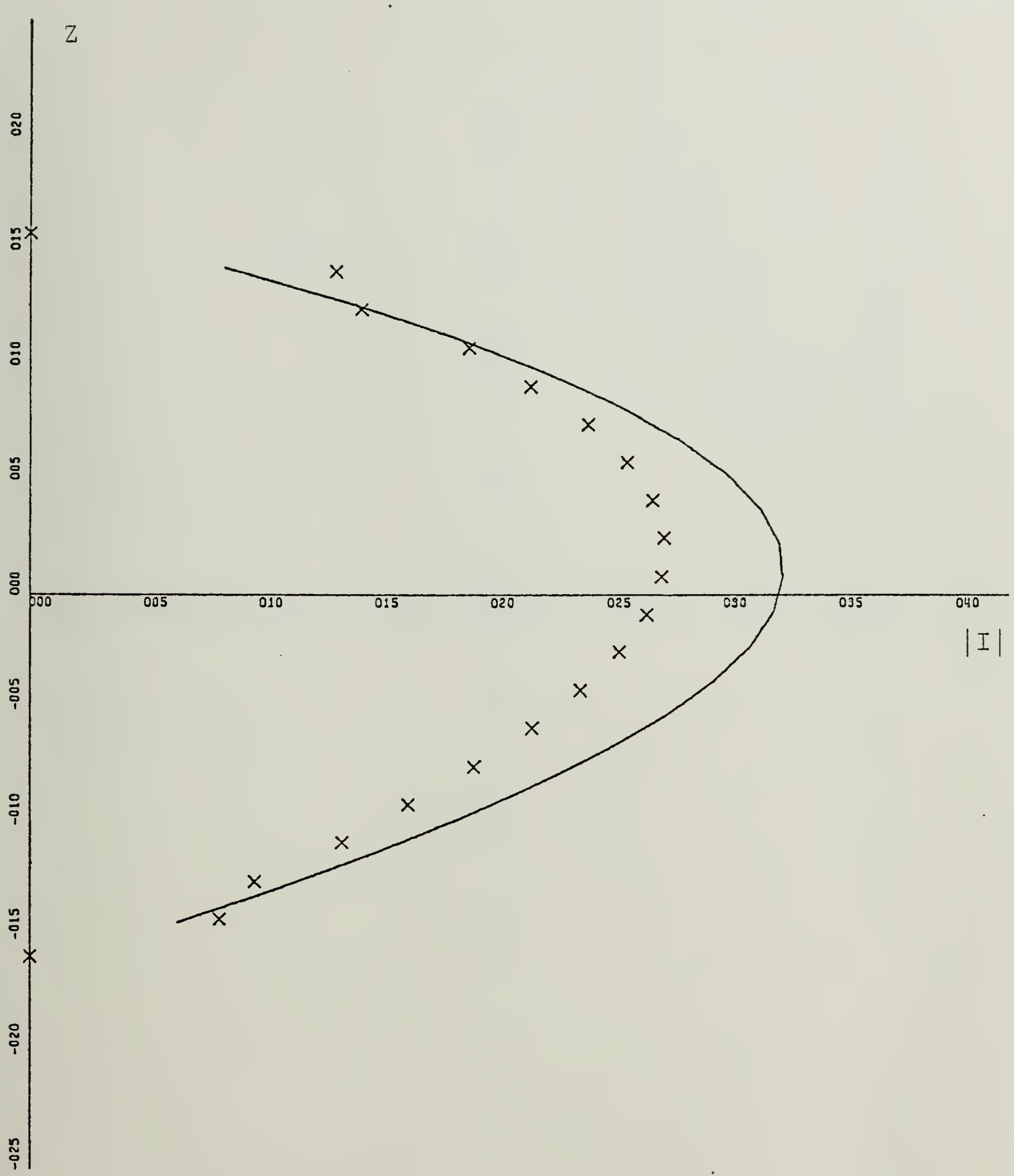
A-2



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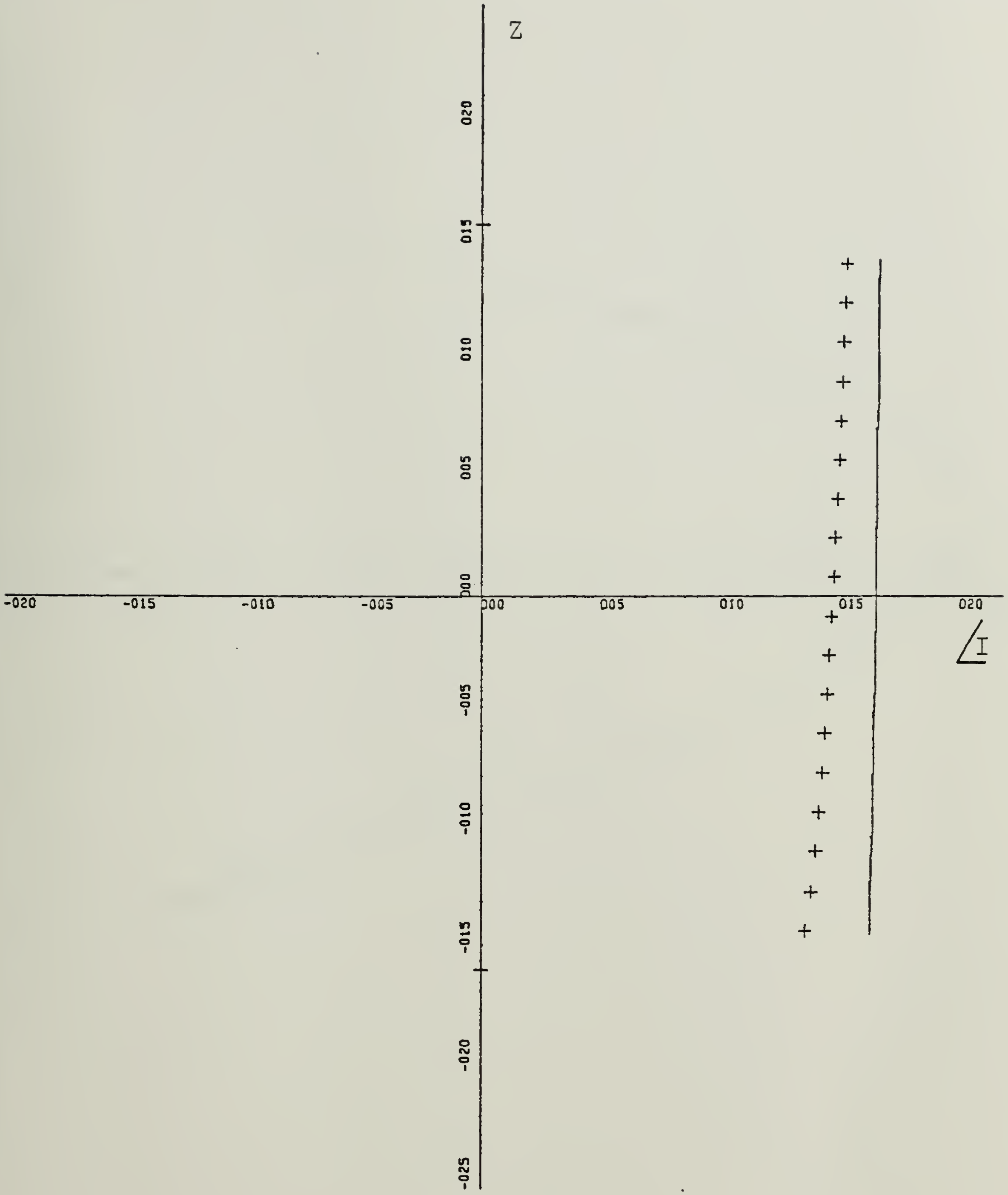
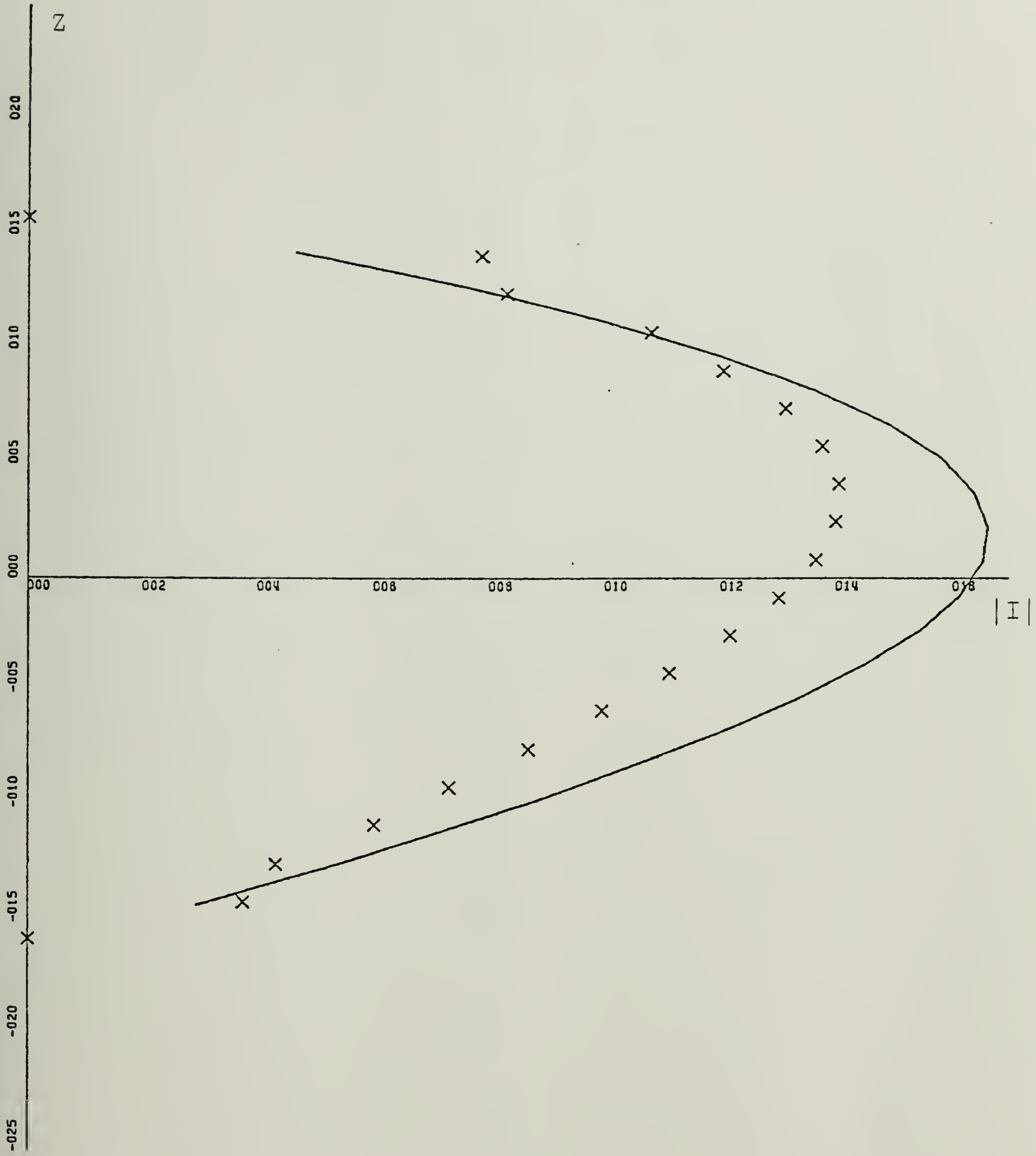
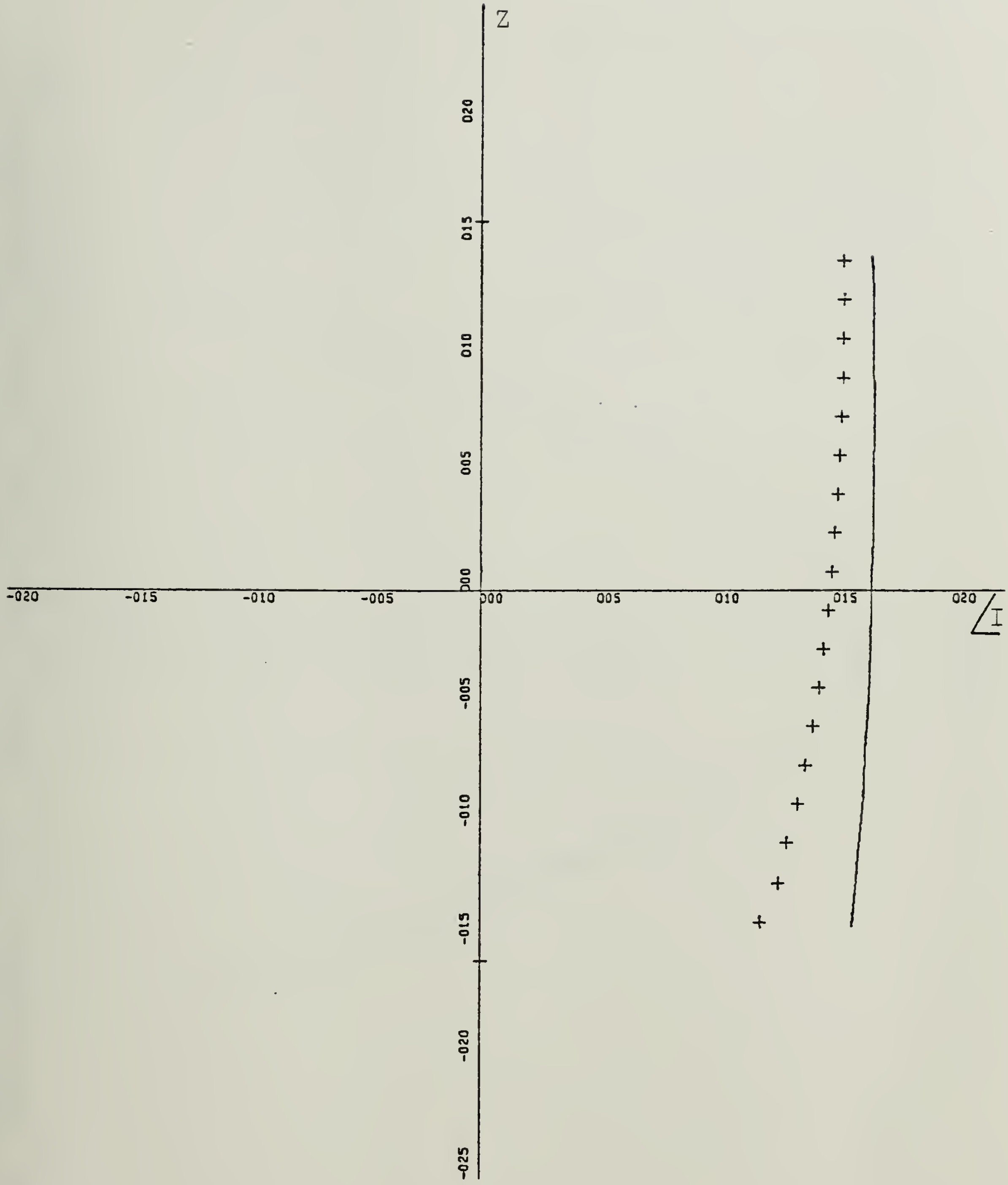


X-SCALE=5.00E+01 UNITS INCH.
Y-SCALE=5.00E-01 UNITS INCH.
CURRENT PHASE :FEM/T=SOLID :INT.EQ=PLUS
ZL=.5:AL=0.02:BL=0.001:ALFA=60:NW=20



X-SCALE=2.00E-04 UNITS INCH.
Y-SCALE=5.00E-01 UNITS INCH.
CURRENT MAGNITUDE :FEM/T=SOLID :INT.EQ=X
ZL=.5:AL=0.02:BL=0.001:ALFA=30:NW=20



X-SCALE=5.00E+01 UNITS INCH.
Y-SCALE=5.00E-01 UNITS INCH.
CURRENT PHASE :FEM/T=SOLID :INT.EQ=PLUS
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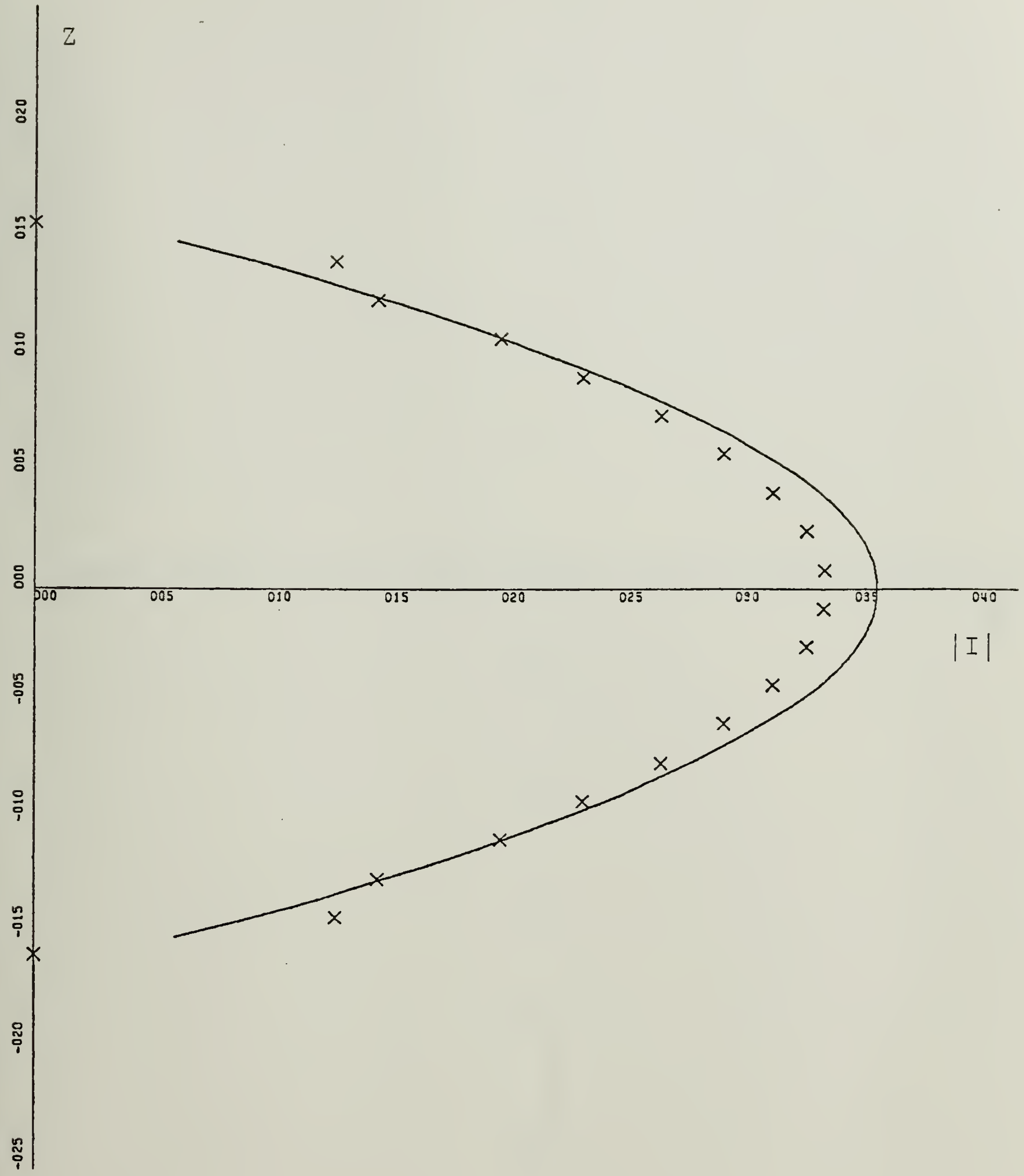

X-SCALE=5.00E-04 UNITS INCH.
Y-SCALE=5.00E-01 UNITS INCH.
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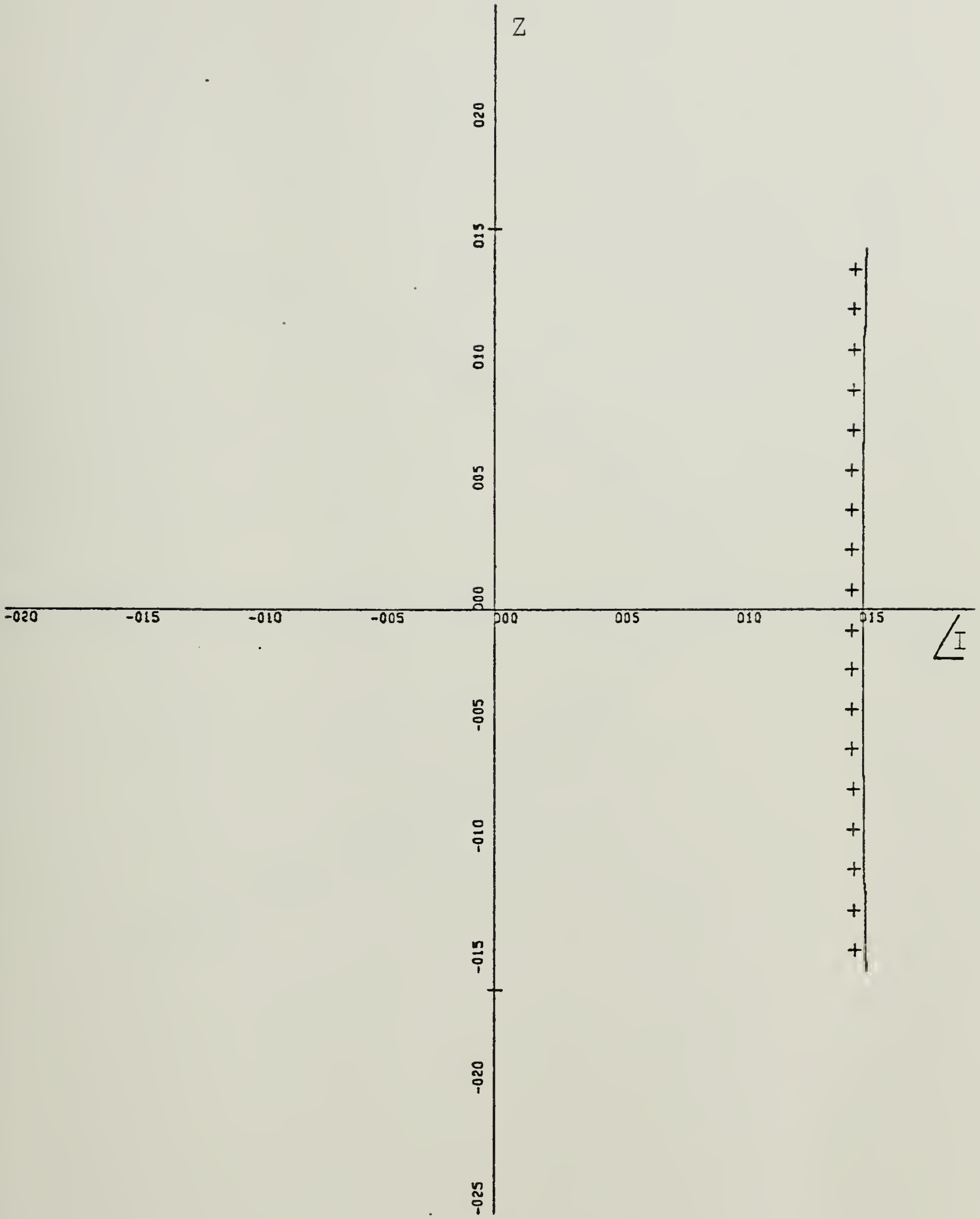
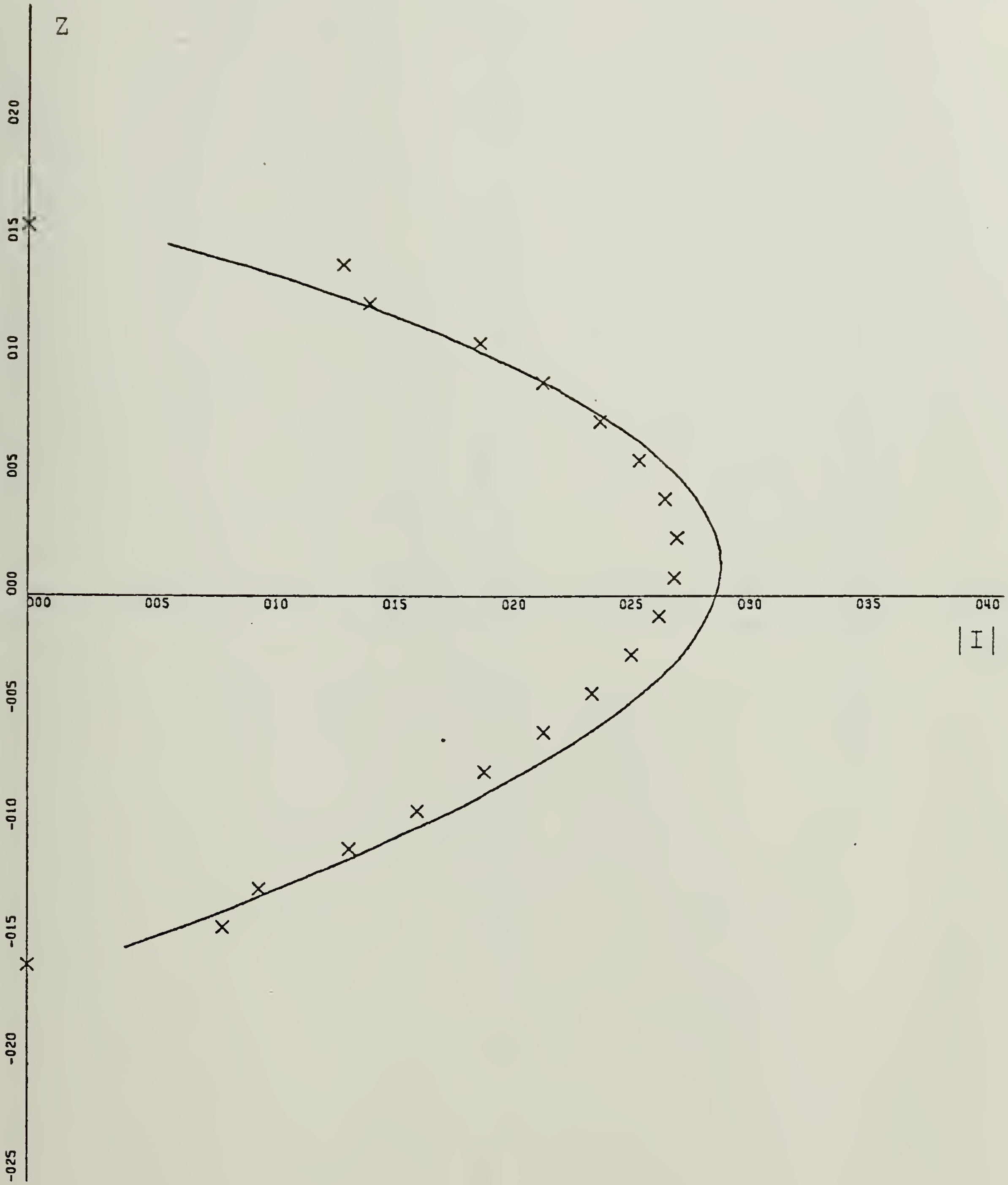
ZL=.5:AL=0.02:BL=0.001:ALFA=90:NW=40

A-7

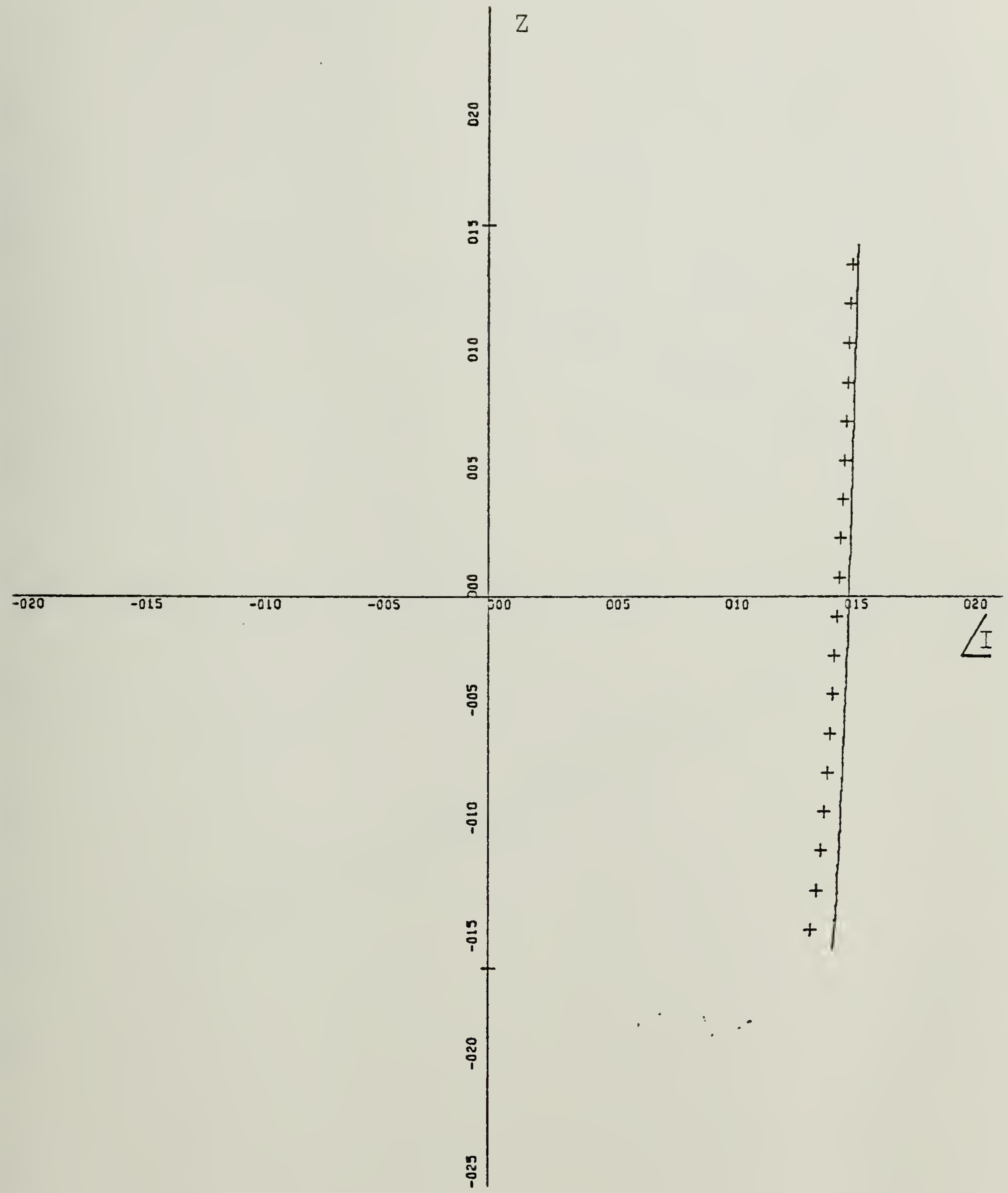


X-SCALE=5.00E+01 UNITS INCH.
Y-SCALE=5.00E-01 UNITS INCH.
CURRENT PHASE :FEM/T=SOLID :INT.EQ=PLUS
ZL=.5:AL=0.02:BL=0.001:ALFA=90:NW=40

A-8

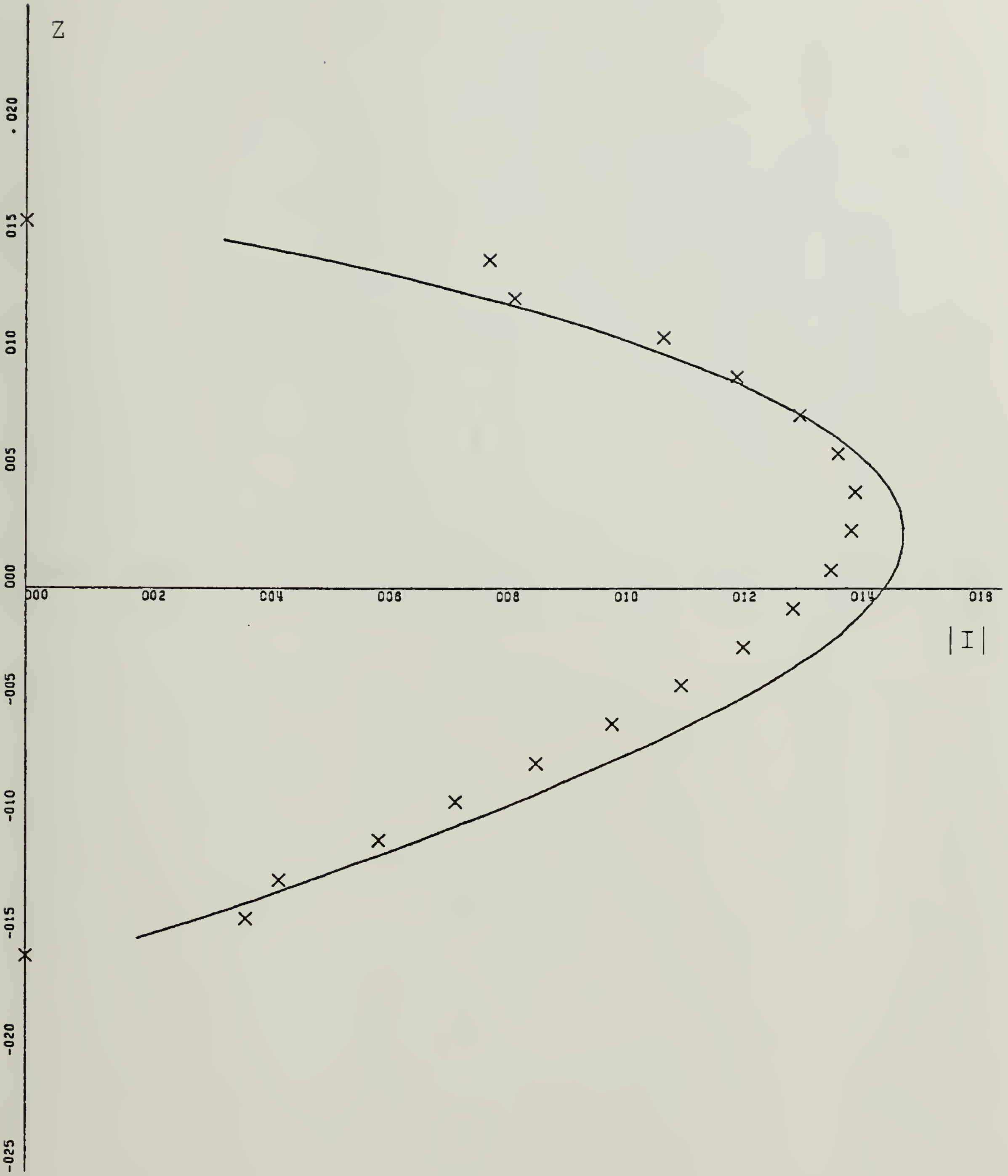


X-SCALE=5.00E-04 UNITS INCH.
Y-SCALE=5.00E-01 UNITS INCH.
CURRENT MAGNITUDE :FEM/T=SØLID :INT.EQ=X
ZL=.5:AL=0.02:BL=0.001:ALFA=60:NW=40



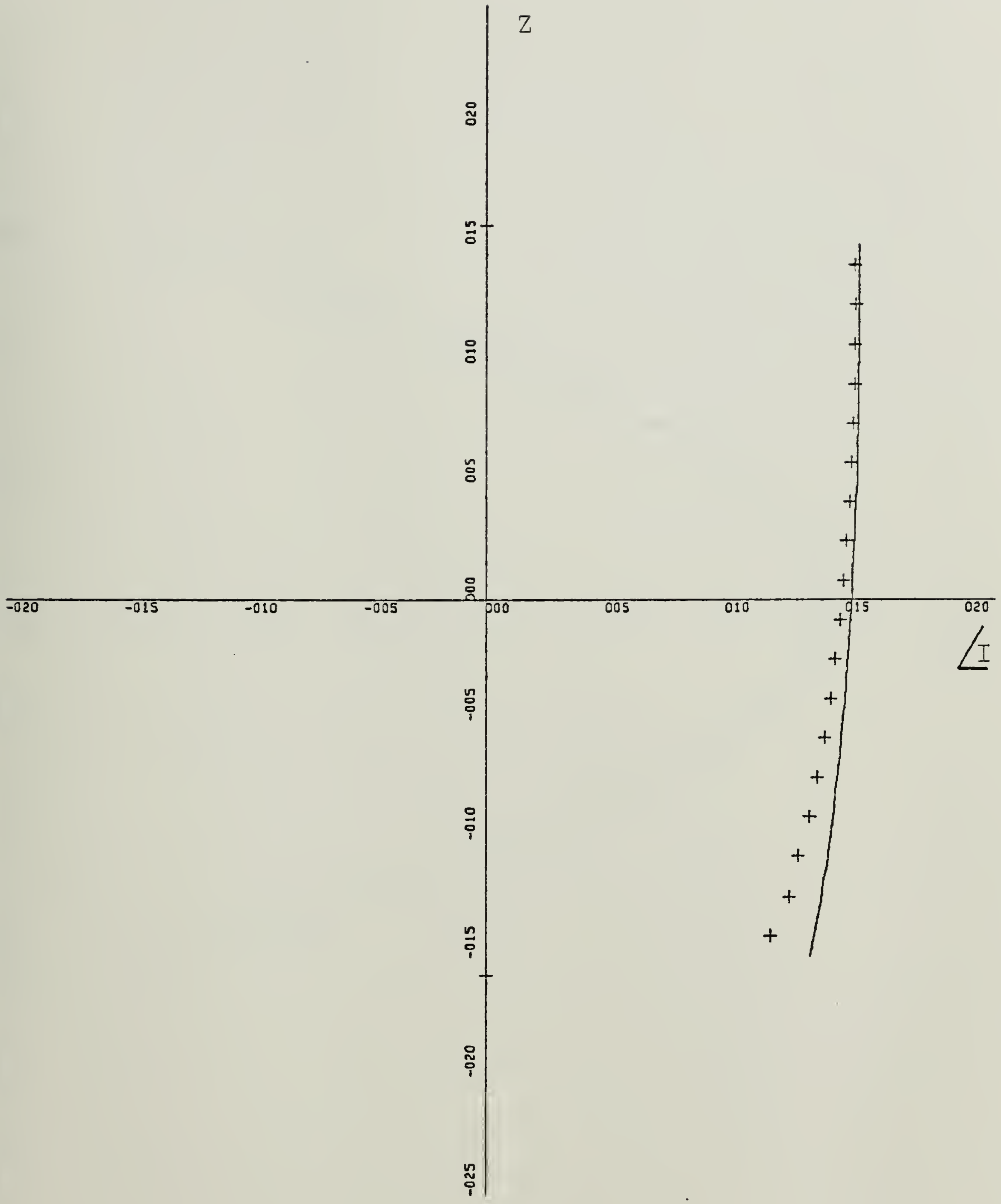
X-SCALE=5.00E+01 UNITS INCH.
Y-SCALE=5.00E-01 UNITS INCH.
CURRENT PHASE :FEM/T=SOLID :INT.EQ=PLUS
ZL=.5:AL=0.02:BL=0.001:ALFA=60:NW=40

A-10

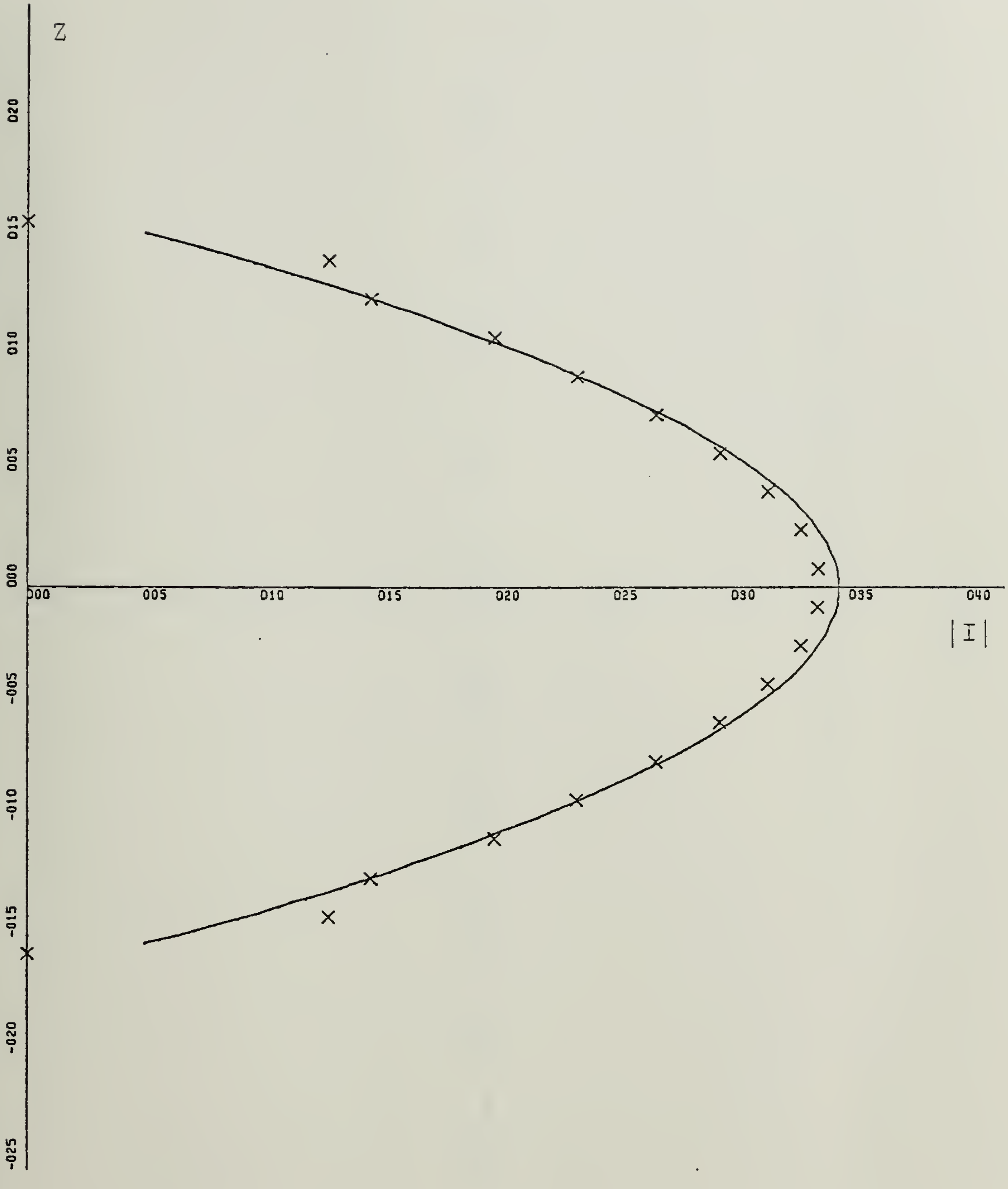


X-SCALE=2.00E-04 UNITS INCH.
Y-SCALE=5.00E-01 UNITS INCH.
CURRENT MAGNITUDE :FEM/T=SØLID :INT.EQ=X
ZL=.5:AL=0.02:BL=0.001:ALFA=30:NW=40

A-11

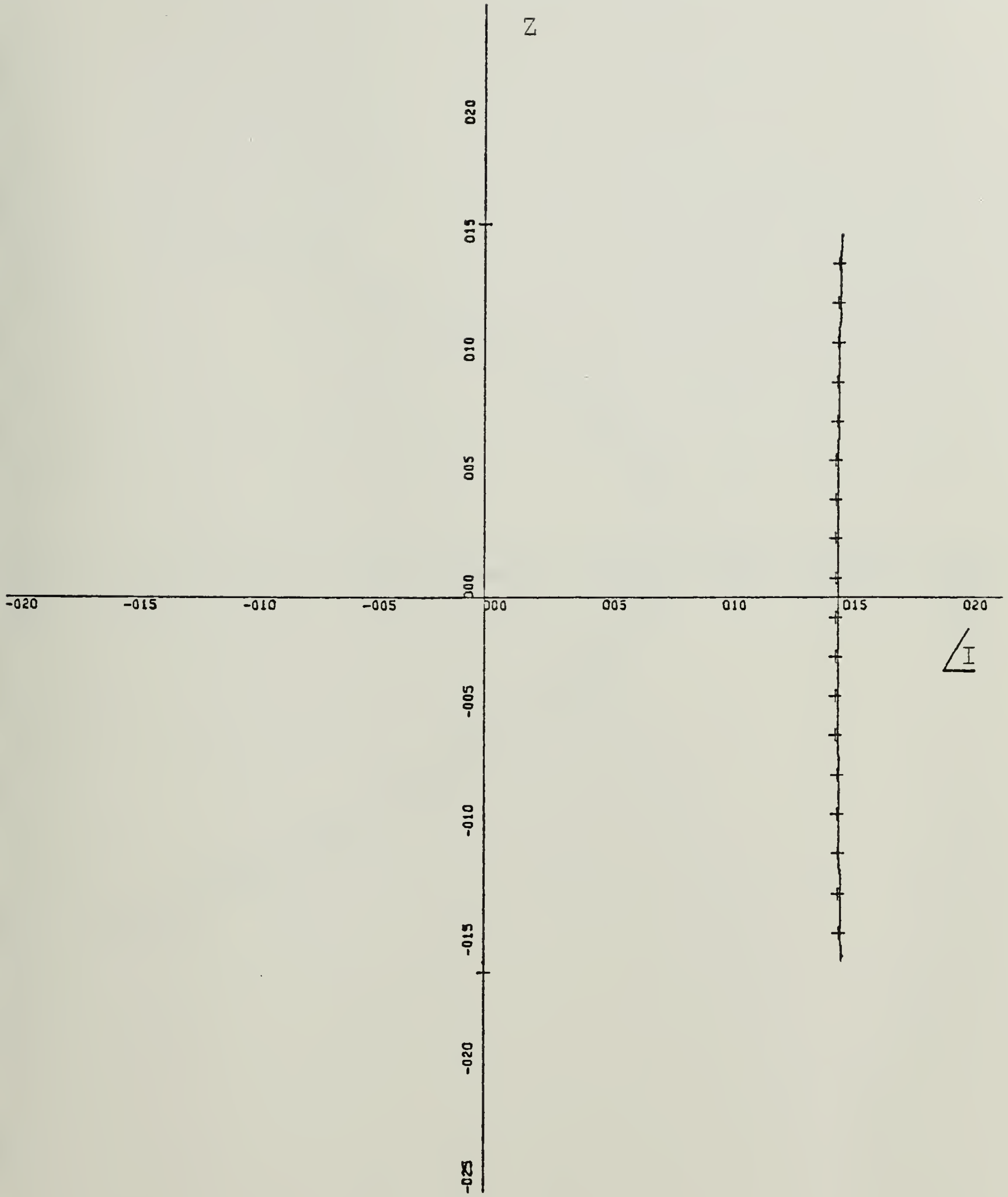


X-SCALE=5.00E+01 UNITS INCH.
Y-SCALE=5.00E-01 UNITS INCH.
CURRENT PHASE :FEM/T=SOLID :INT.EQ=PLUS
ZL=.5:AL=0.02:BL=0.001:ALFA=30:NW=40



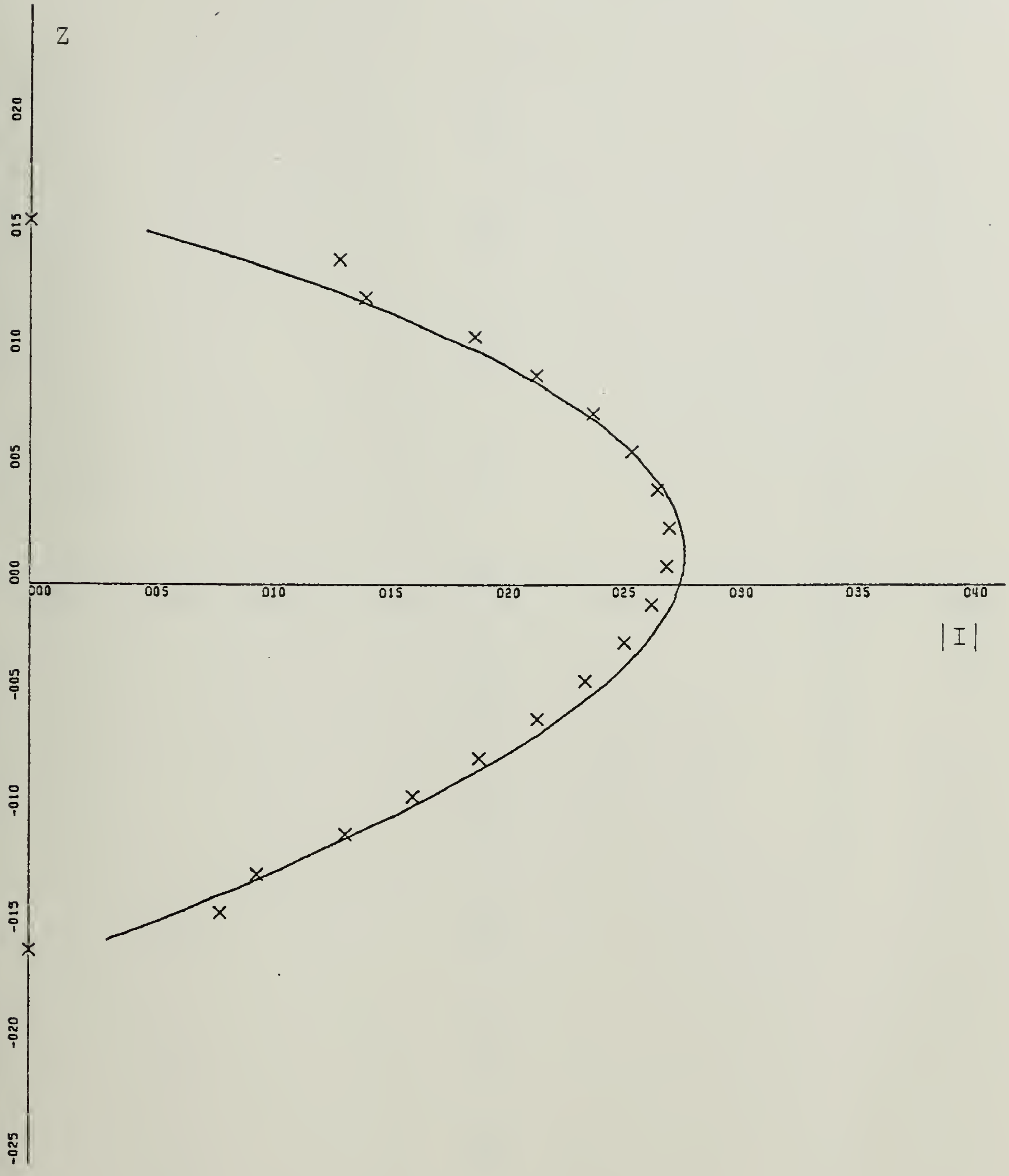
X-SCALE=5.00E-04 UNITS INCH.
Y-SCALE=5.00E-01 UNITS INCH.
CURRENT MAGNITUDE :FEM/T=SOLID :INT.EQ=X
ZL=.5:AL=0.02:BL=0.001:ALFA=90:NW=70

A-13



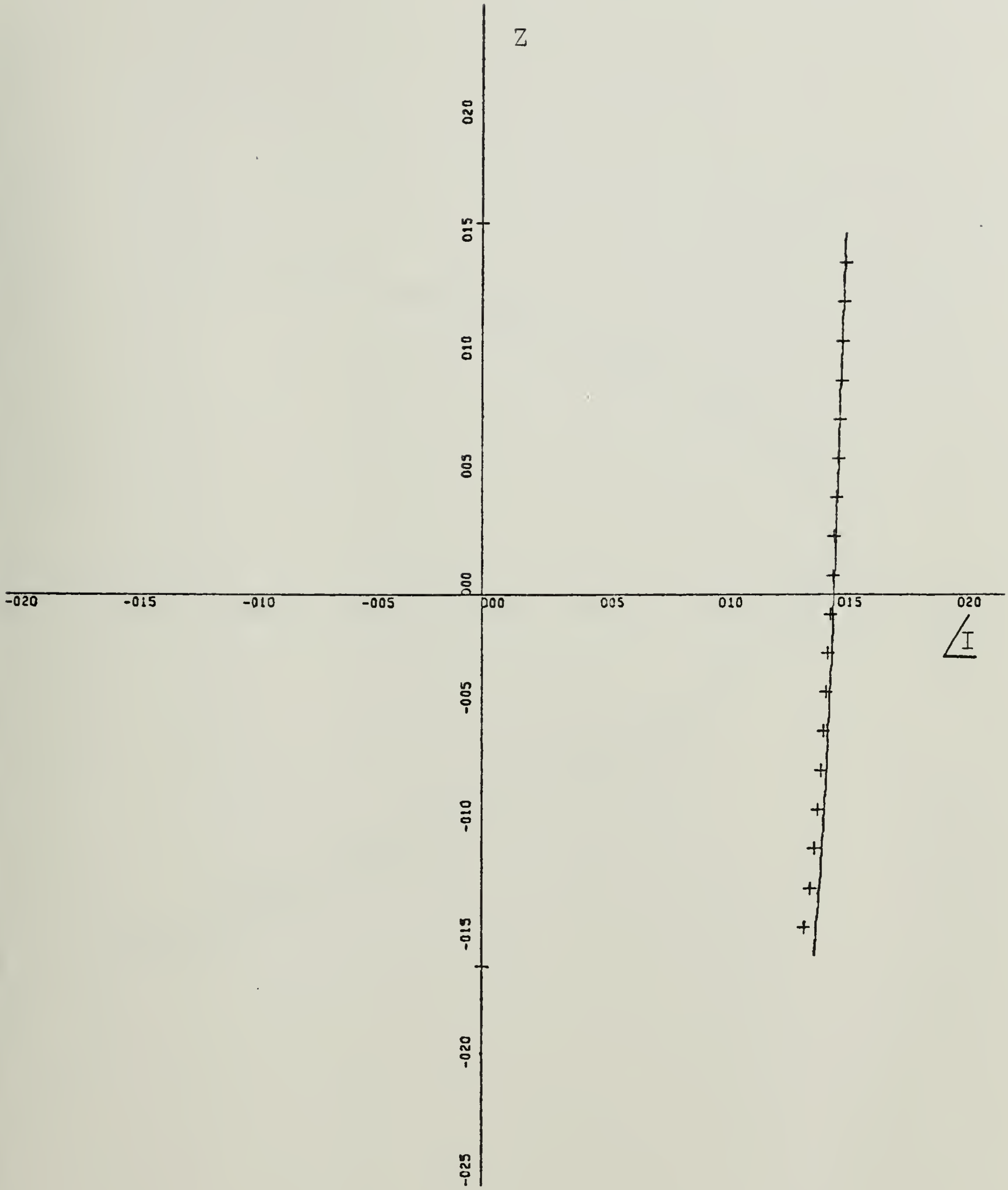
X-SCALE=5.00E+01 UNITS INCH.
Y-SCALE=5.00E-01 UNITS INCH.
CURRENT PHASE :FEM/T=SOLID :INT.EQ=PLUS
ZL=.5:AL=0.02:BL=0.001:ALFA=90:NW=70

A-14



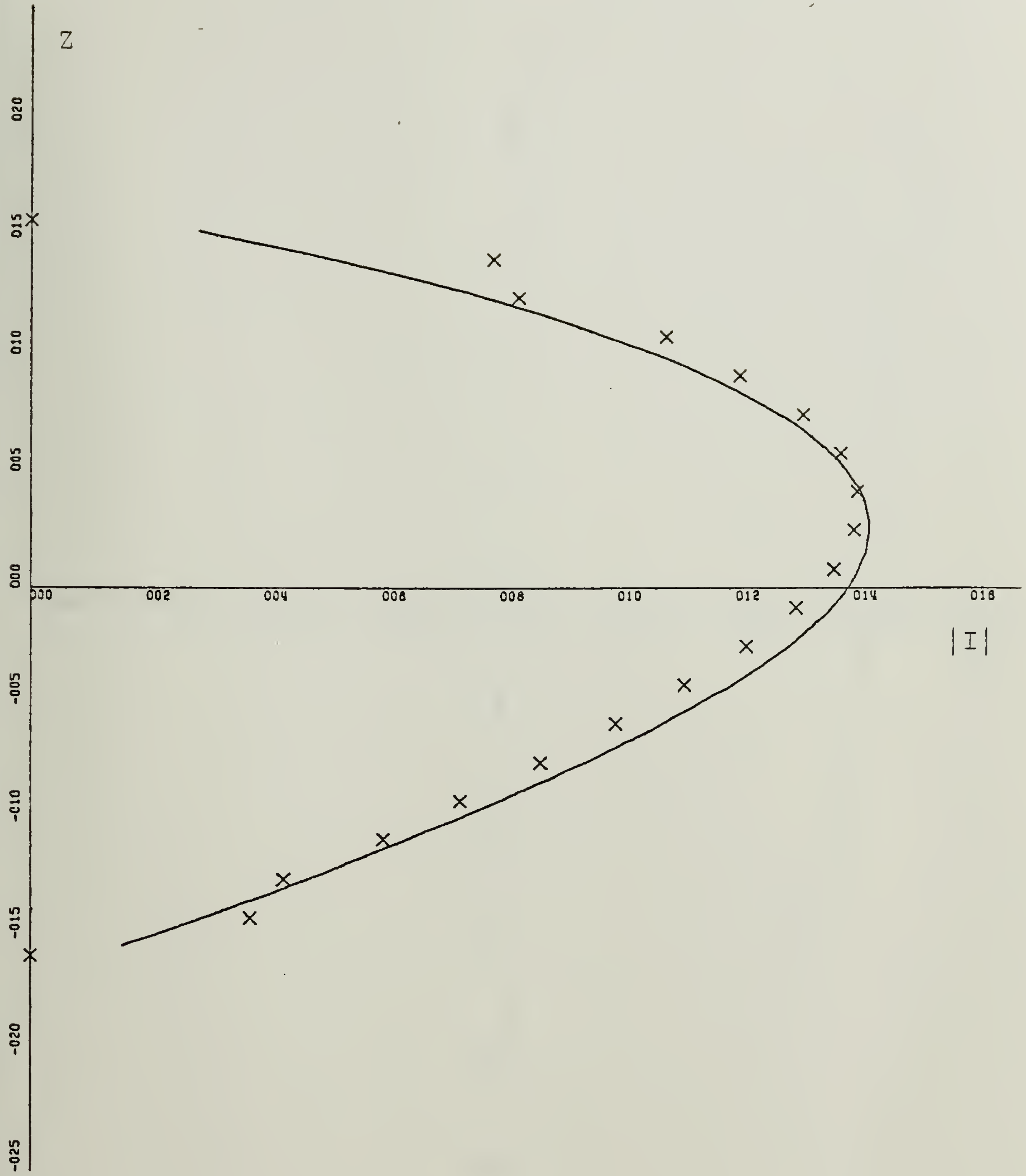
X-SCALE=5.00E-04 UNITS INCH.
Y-SCALE=5.00E-01 UNITS INCH.
CURRENT MAGNITUDE :FEM/T=SOLID :INT.EQ=X
ZL=.5:AL=0.02:BL=0.001:ALFA=60:NW=70

A-15

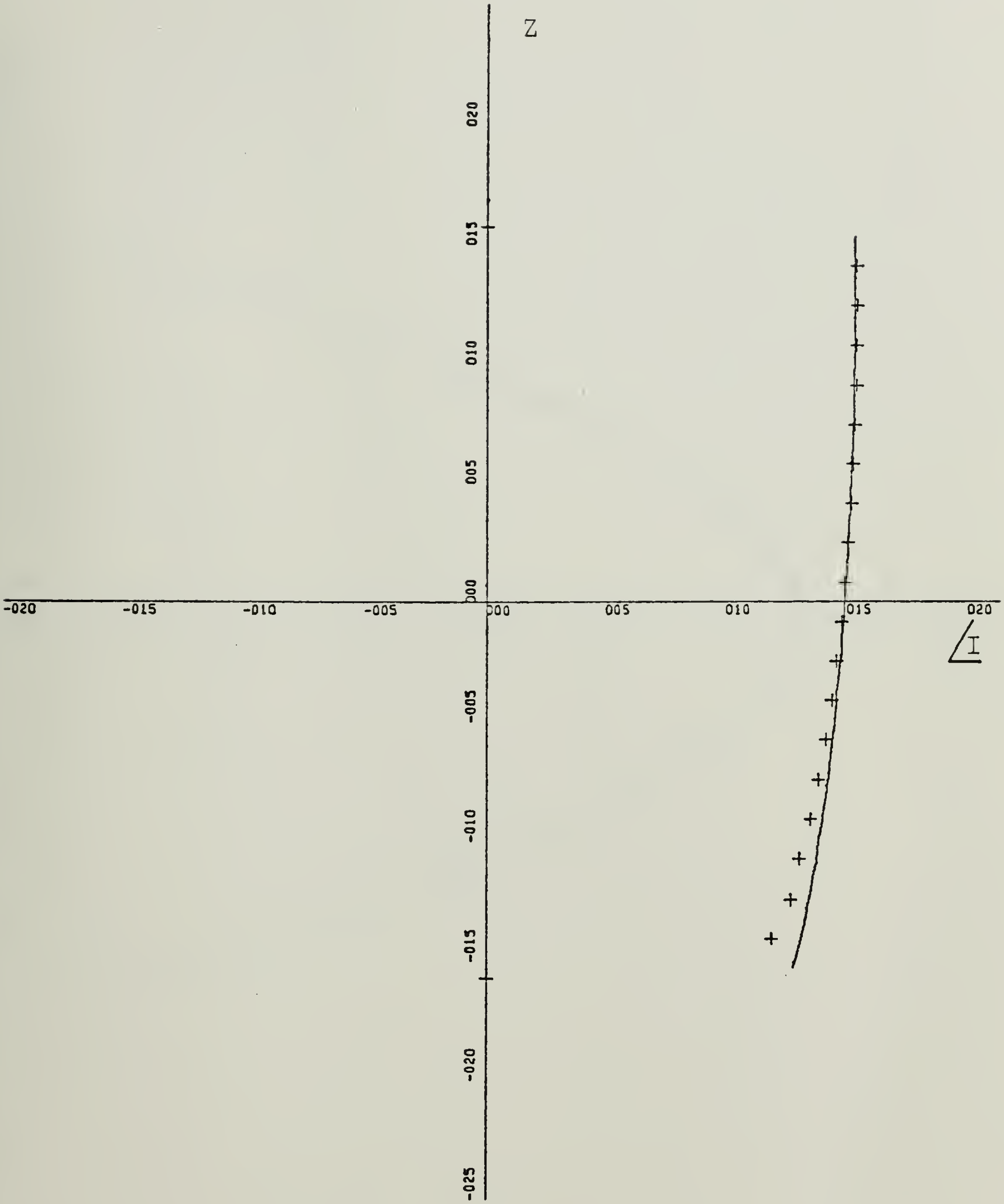


X-SCALE=5.00E+01 UNITS INCH.
Y-SCALE=5.00E-01 UNITS INCH.
CURRENT PHASE :FEM/T=SOLID :INT.EQ=PLUS
ZL=.5:AL=0.02:BL=0.001:ALFA=60:NW=70

A-16

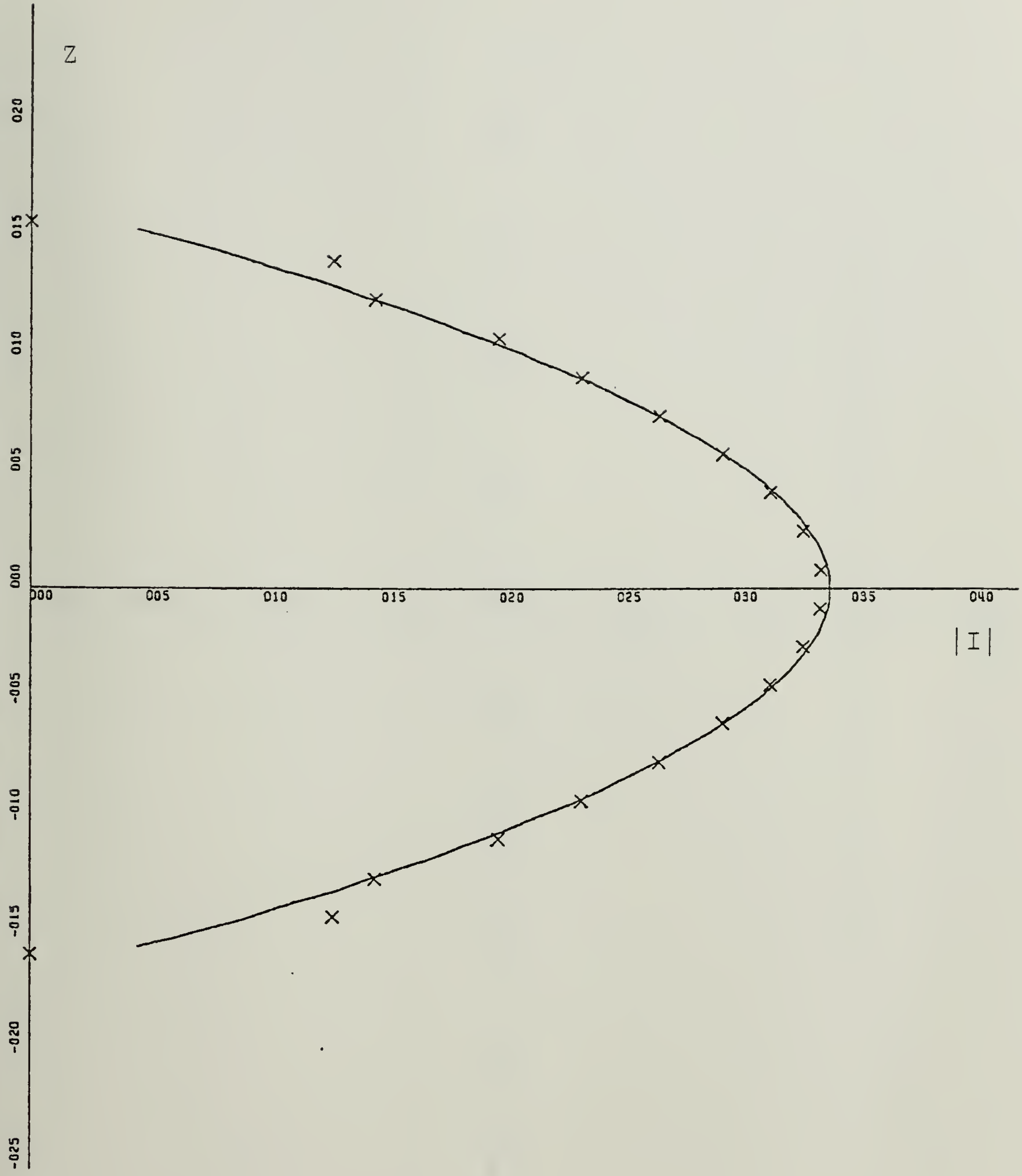


X-SCALE=2.00E-04 UNITS INCH.
Y-SCALE=5.00E-01 UNITS INCH.
CURRENT MAGNITUDE :FEM/T=SOLID :INT.EQ=X
ZL=.5:AL=0.02:BL=0.001:ALFA=30:NW=70

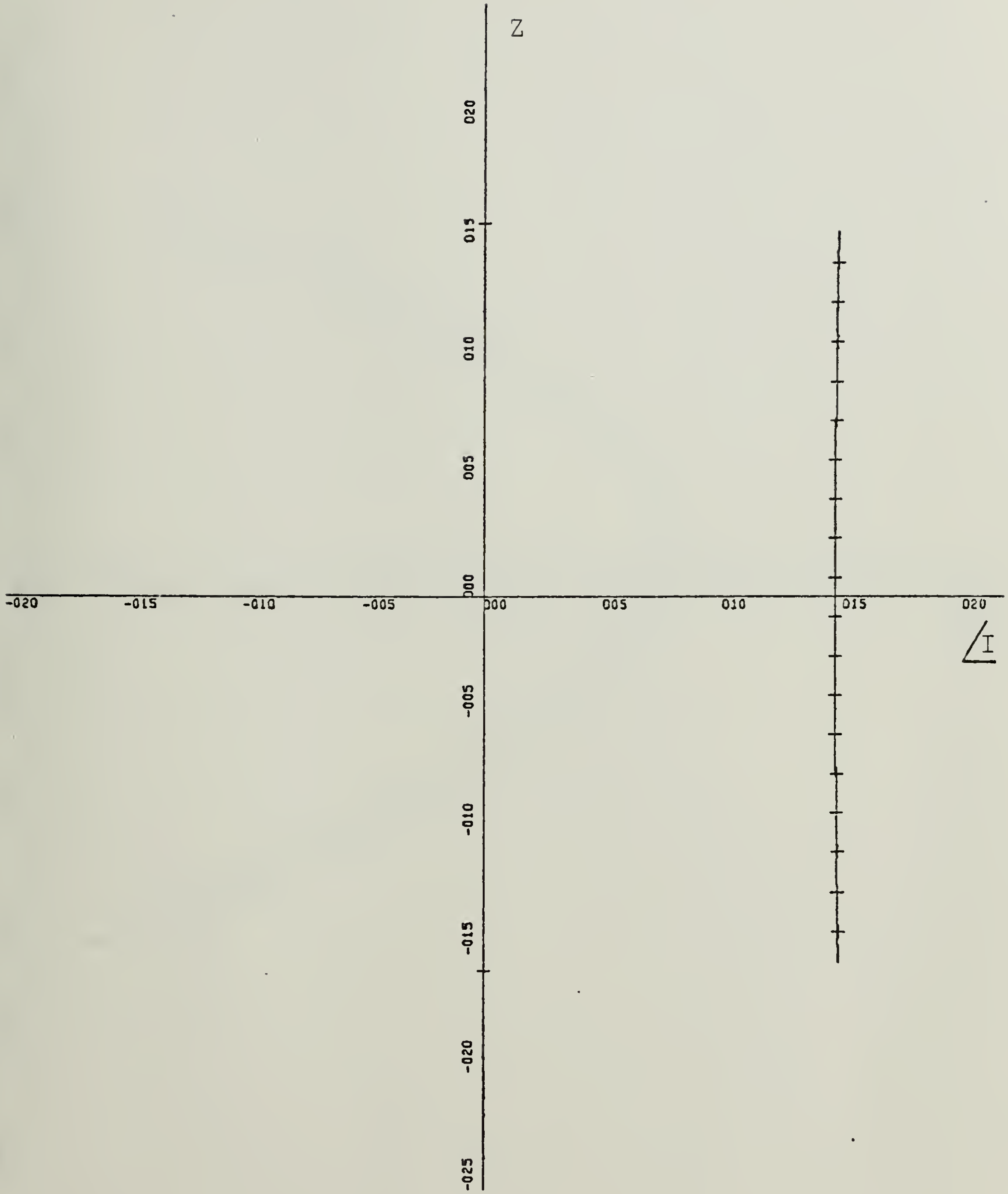


X-SCALE=5.00E+01 UNITS INCH.
Y-SCALE=5.00E-01 UNITS INCH.
CURRENT PHASE :FEM/T=SOLID :INT.EQ=PLUS
ZL=.5:AL=0.02:BL=0.001:ALFA=30:NW=70

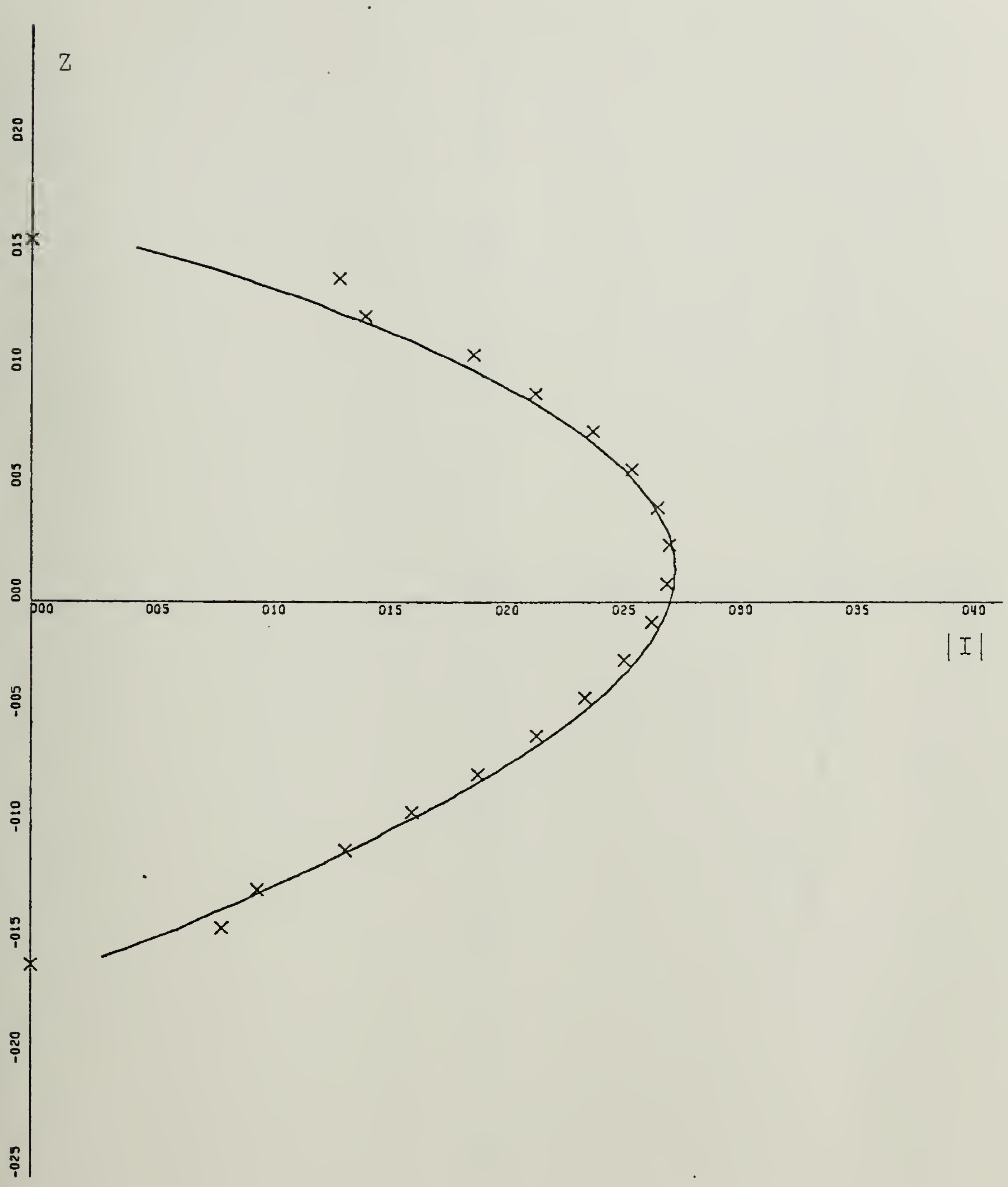
A-18



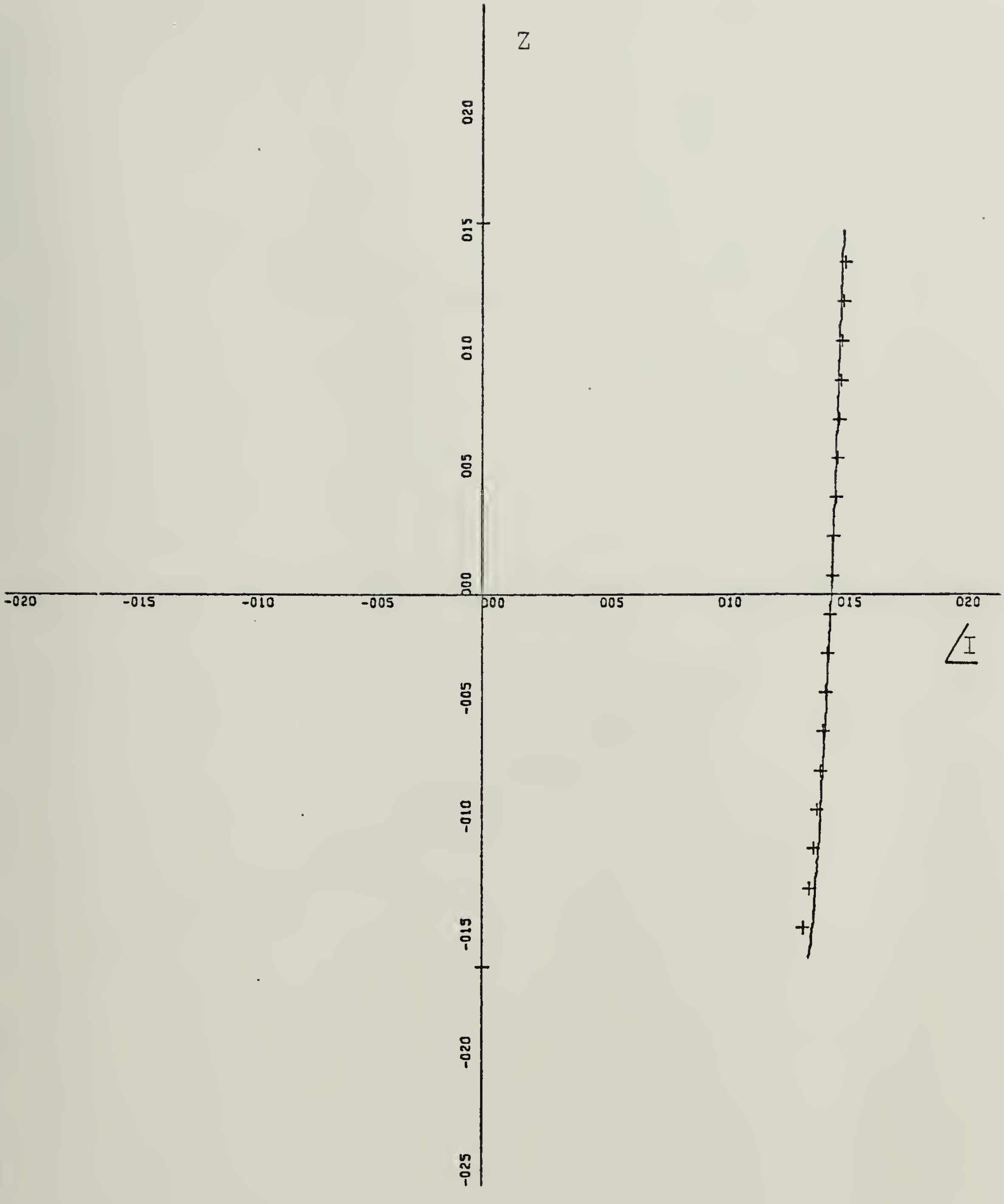
X-SCALE=5.00E-04 UNITS INCH.
Y-SCALE=5.00E-01 UNITS INCH.
CURRENT MAGNITUDE :FEM/T=SOLID :INT.EQ=X
ZL=0.5:AL=0.02:BL=0.001:ALFA=90:NW=100



X-SCALE=5.00E+01 UNITS INCH.
Y-SCALE=5.00E-01 UNITS INCH.
CURRENT PHASE :FEM/T=SOLID :INT.EQ=PLUS
ZL=0.5:AL=0.02:BL=0.001:ALFA=90:NW=100



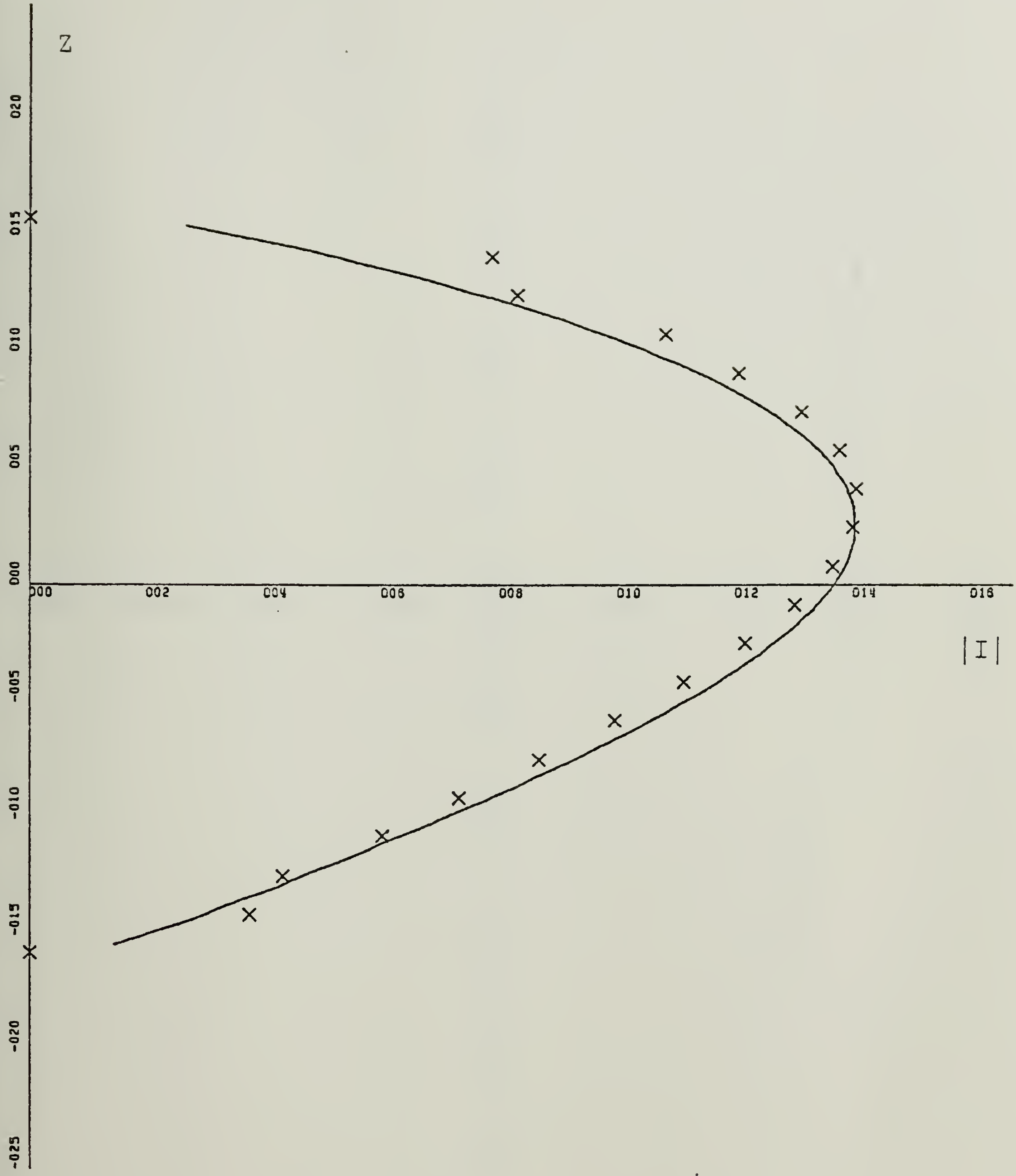
X-SCALE=5.00E-04 UNITS INCH.
Y-SCALE=5.00E-01 UNITS INCH.
CURRENT MAGNITUDE :FEM/T=SOLID :INT.EQ=X
ZL=0.5:AL=0.02:BL=0.001:ALFA=60:NW=100



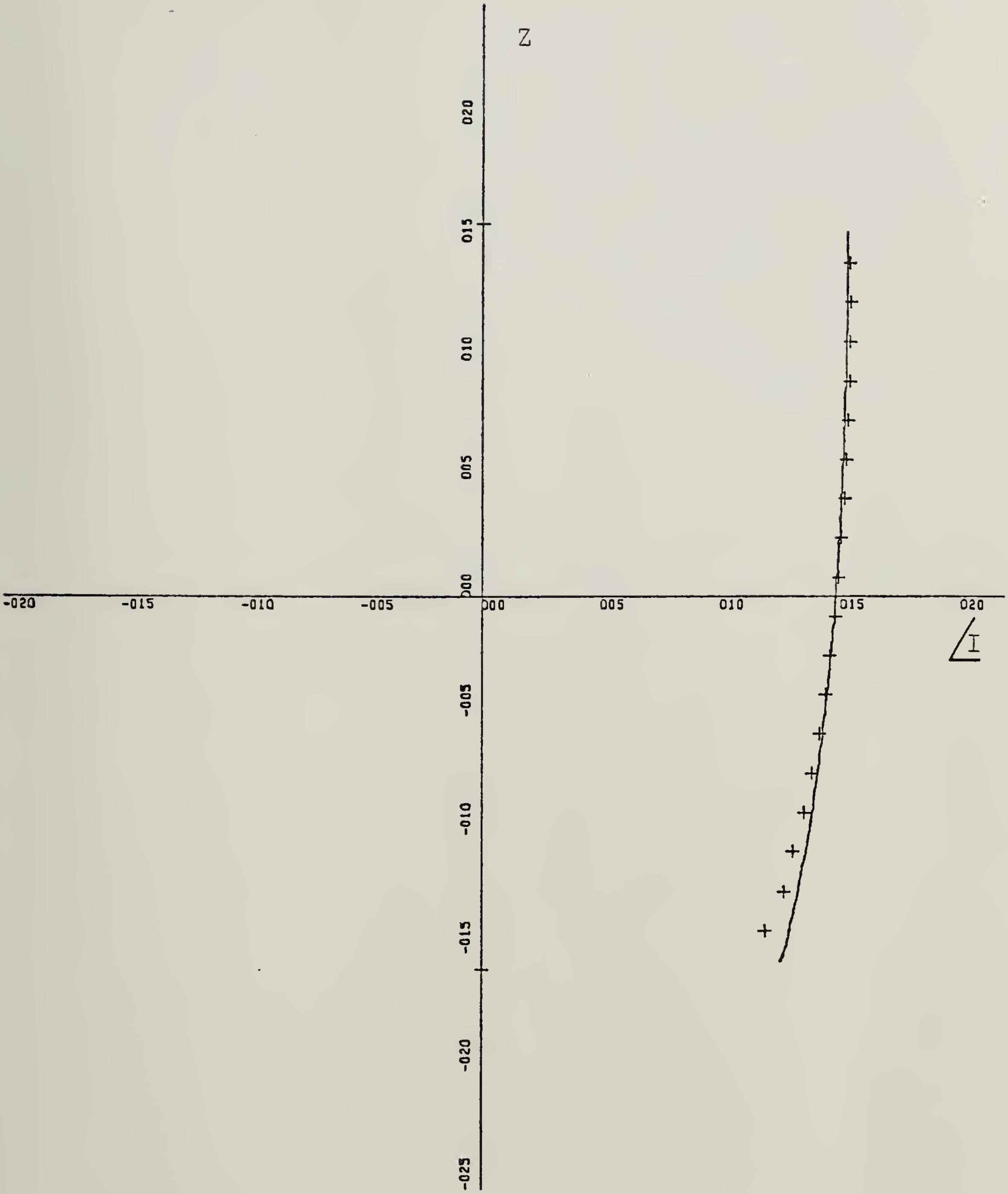
X-SCALE=5.00E+01 UNITS INCH.
Y-SCALE=5.00E-01 UNITS INCH.

CURRENT PHASE :FEM/T=SOLID :INT.EQ=PLUS
ZL=0.5:AL=0.02:BL=0.001:ALFA=60:NW=100

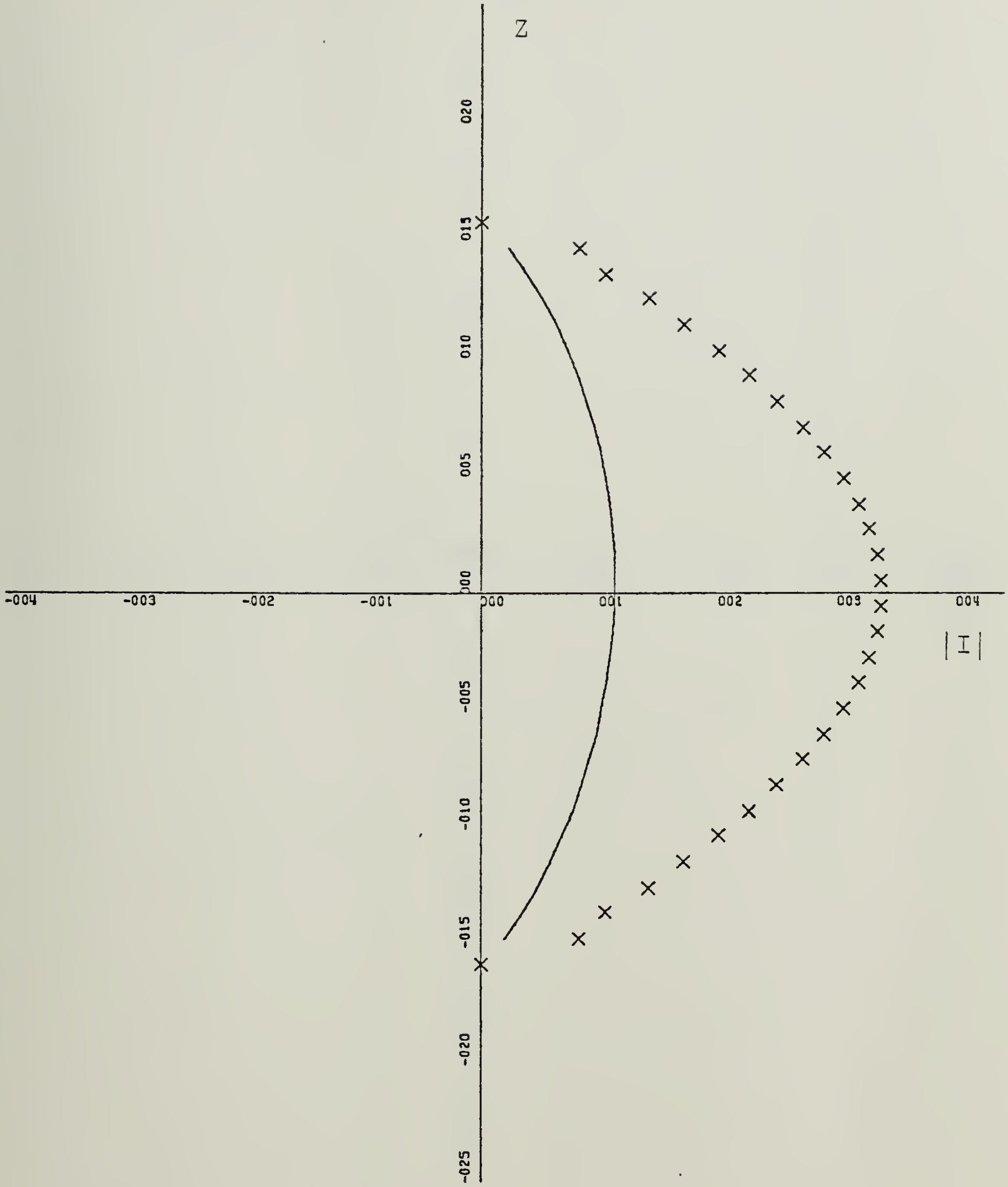
A-22



X-SCALE=2.00E-04 UNITS INCH.
Y-SCALE=5.00E-01 UNITS INCH.
CURRENT MAGNITUDE :FEM/T=SOLID :INT.EQ=X
ZL=0.5:AL=0.02:BL=0.001:ALFA=30:NW=100

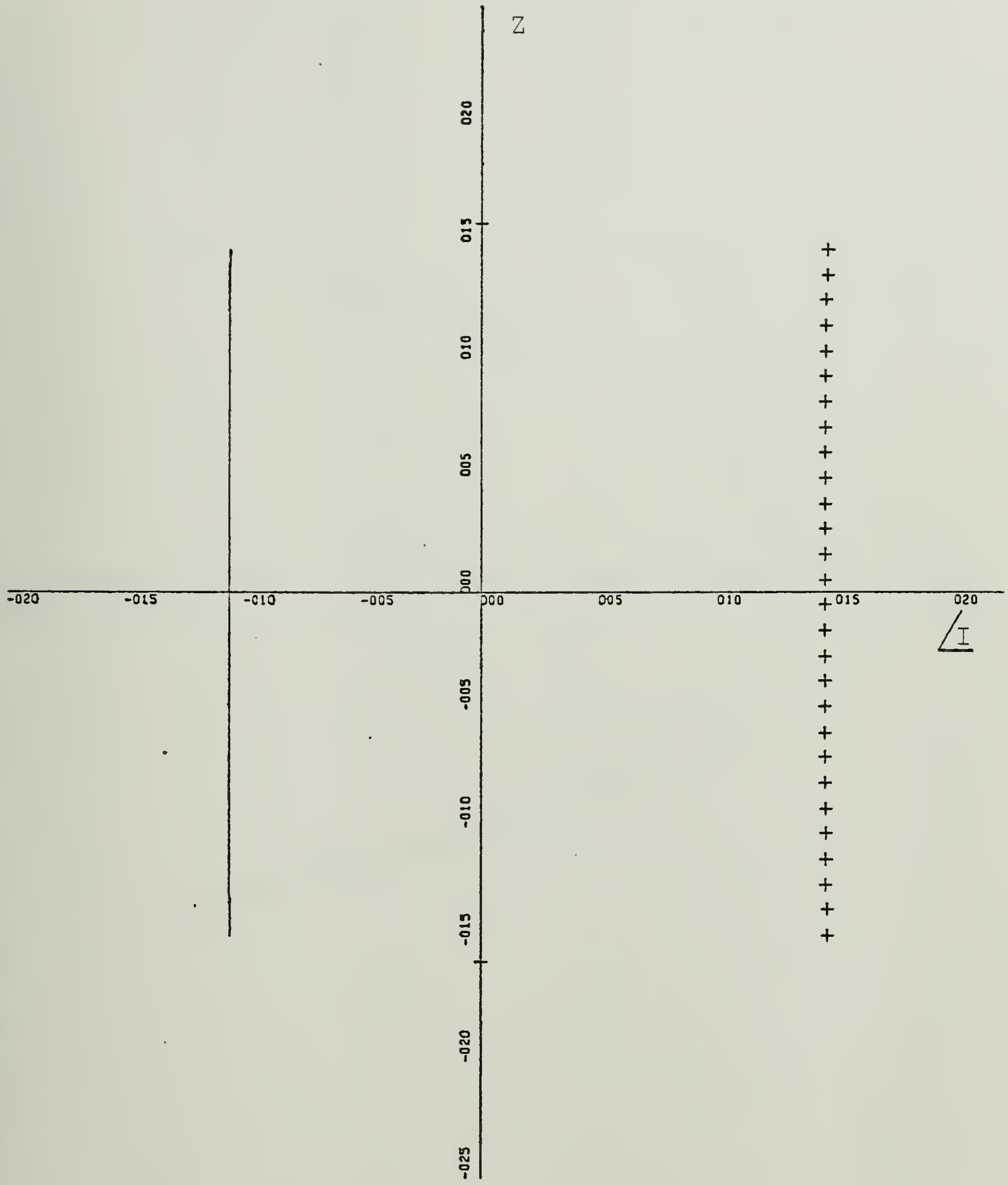


X-SCALE=5.00E+01 UNITS INCH.
Y-SCALE=5.00E-01 UNITS INCH.
CURRENT PHASE :FEM/T=SOLID :INT.EQ=PLUS
ZL=0.5:AL=0.02:BL=0.001:ALFA=30:NW=100

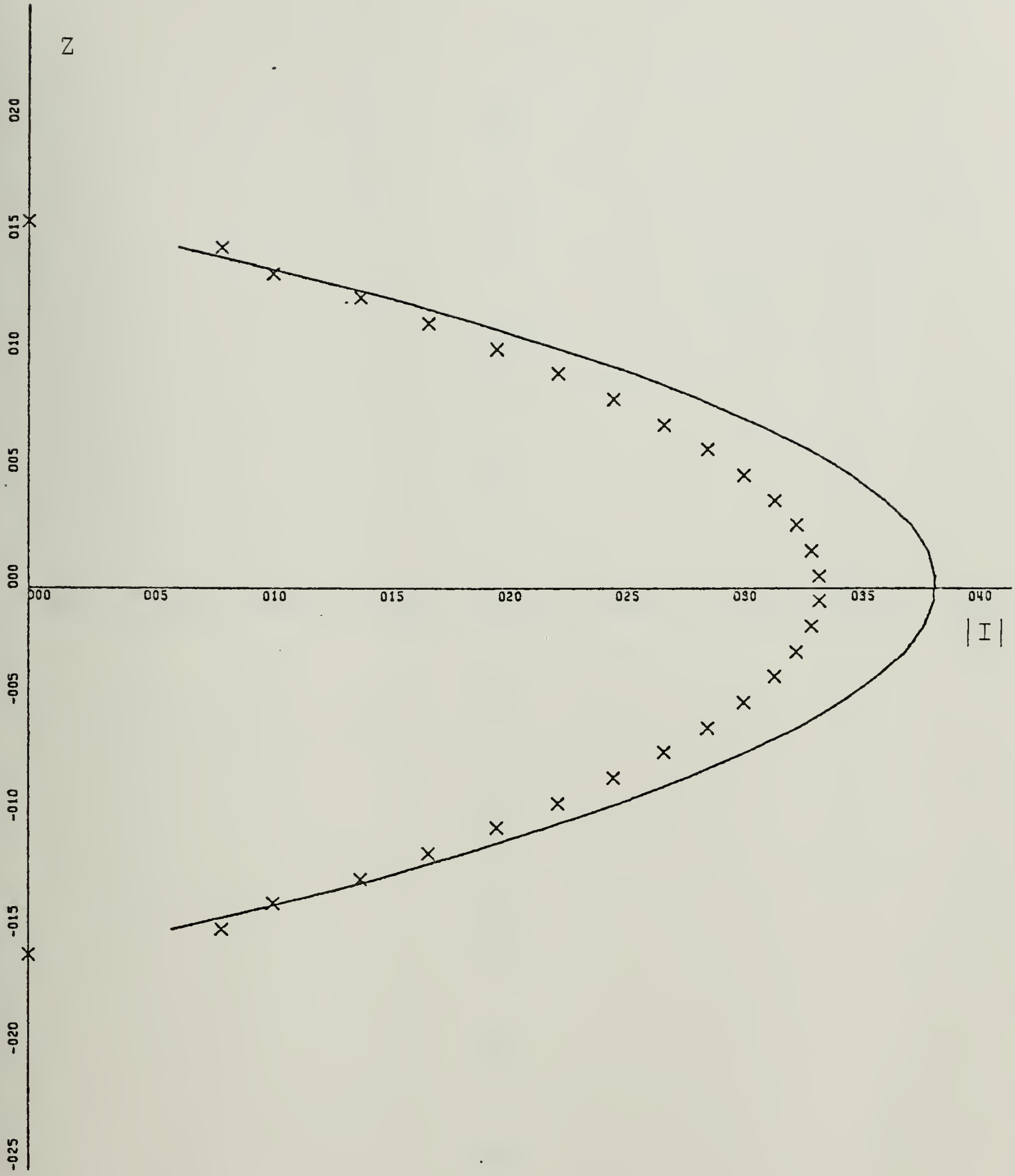


X-SCALE=1.00E-03 UNITS INCH.
Y-SCALE=5.00E-01 UNITS INCH.

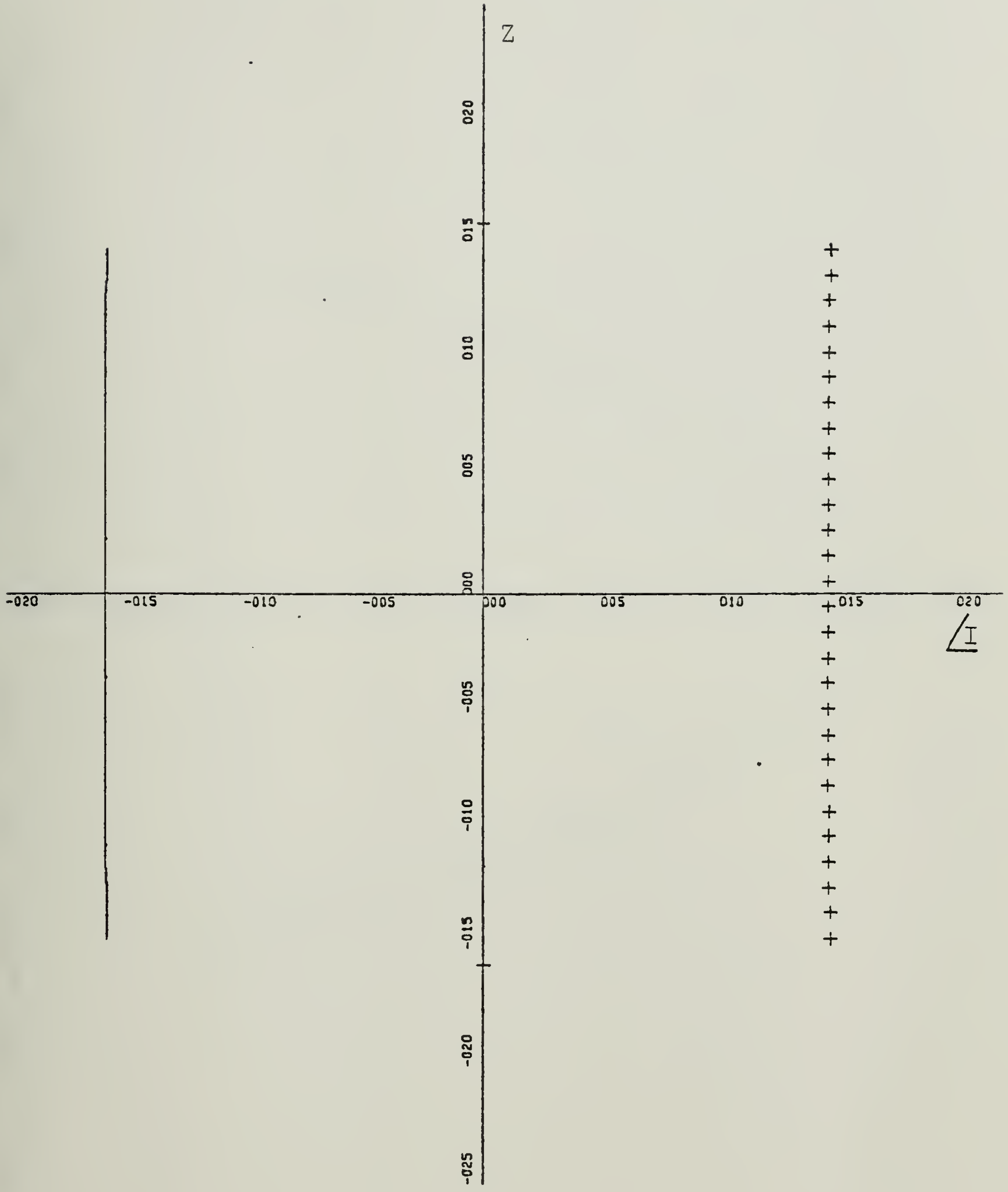
CURRENT MAGNITUDE :FEM/T=SOLID :INT.EQ=X
ZL=.5 : AL=1/100 : DR=DZ/4 : ALFA=90



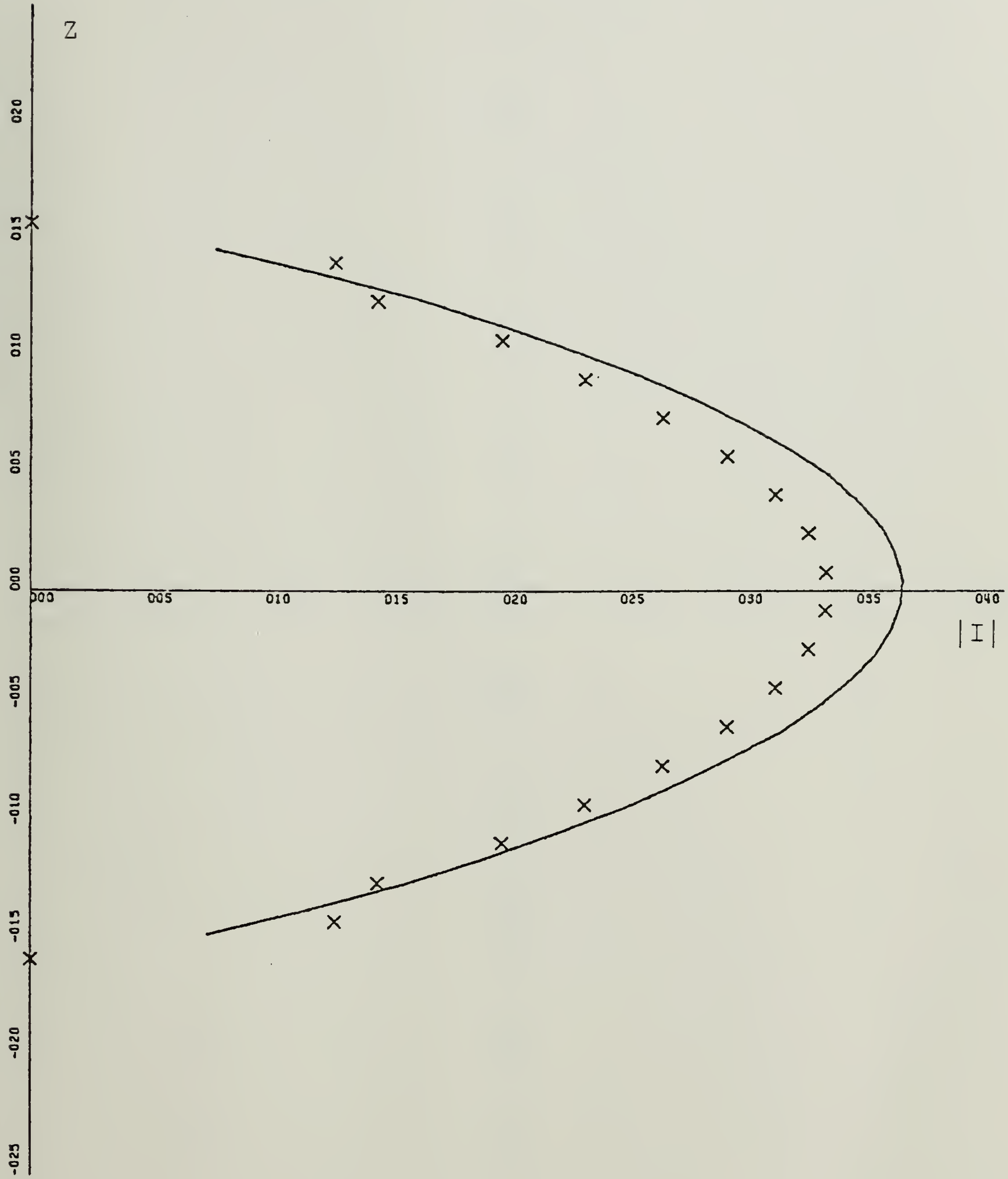
X-SCALE=5.00E+01 UNITS INCH.
Y-SCALE=5.00E-01 UNITS INCH.
CURRENT PHASE :FEM/T=SOLID :INT.EQ=PLUS
ZL=.5 : AL=1/100 : DR=DZ/4 : ALFA=90



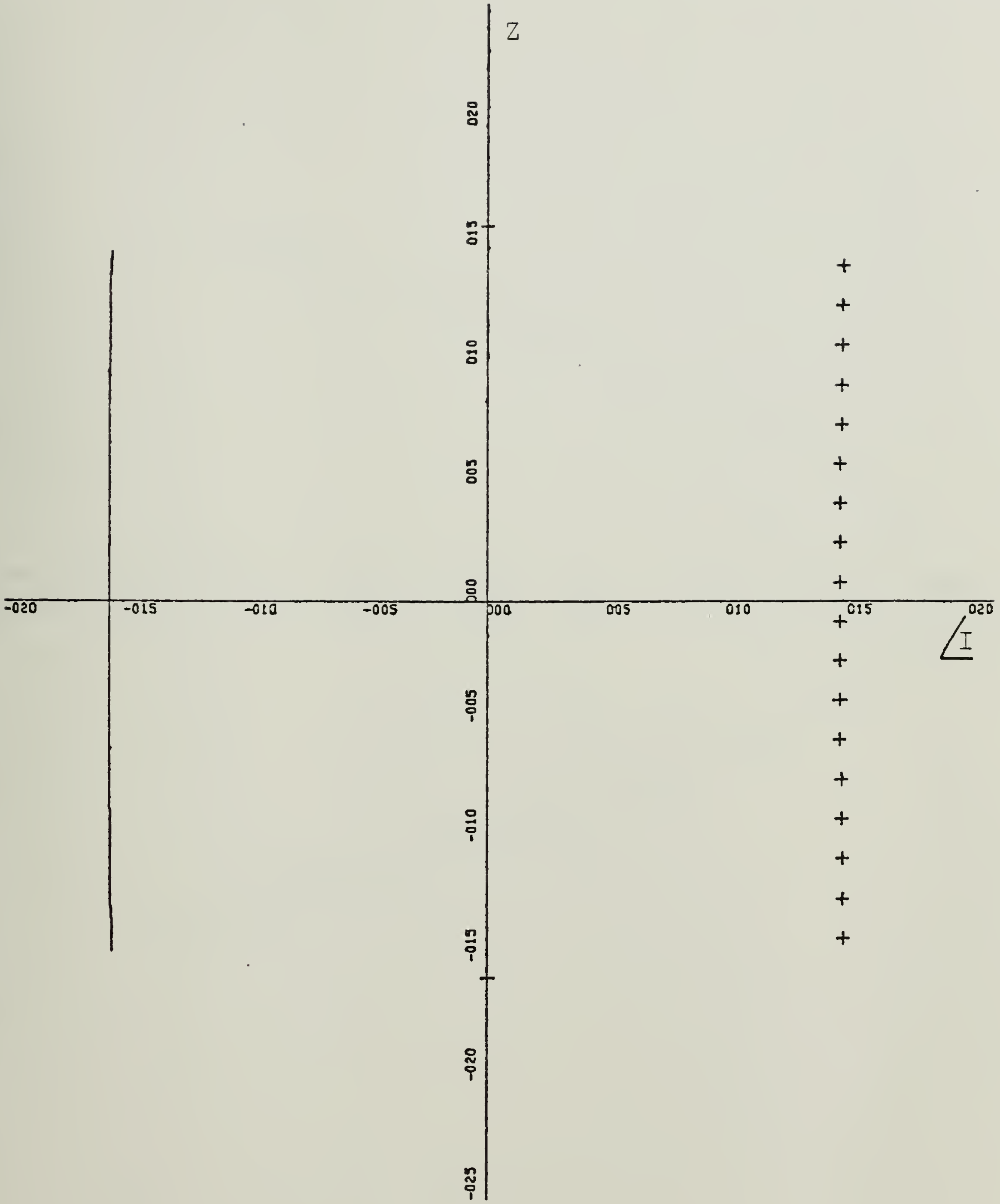
X-SCALE=5.00E-04 UNITS INCH.
Y-SCALE=5.00E-01 UNITS INCH.
CURRENT MAGNITUDE :FEM/T=SOLID :INT.EQ=X
ZL=.5 : AL=1/100 : DR=DZ/9.8 : ALFA=90



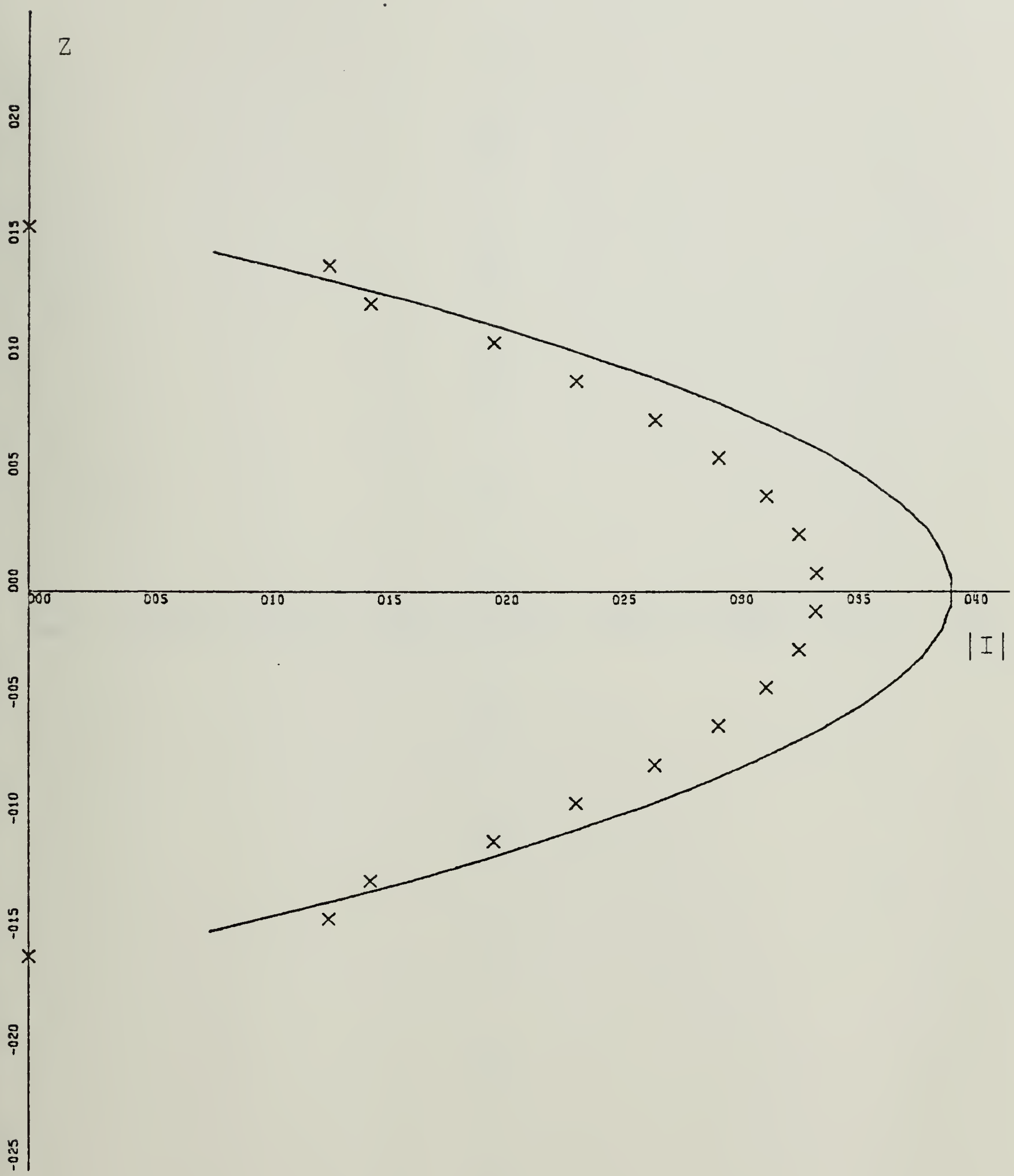
X-SCALE=5.00E+01 UNITS INCH.
Y-SCALE=5.00E-01 UNITS INCH.
CURRENT PHASE :FEM/T=SOLID :INT.EQ=PLUS
ZL=.5 : AL=1/100 : DR=DZ/9.8 : ALFA=90



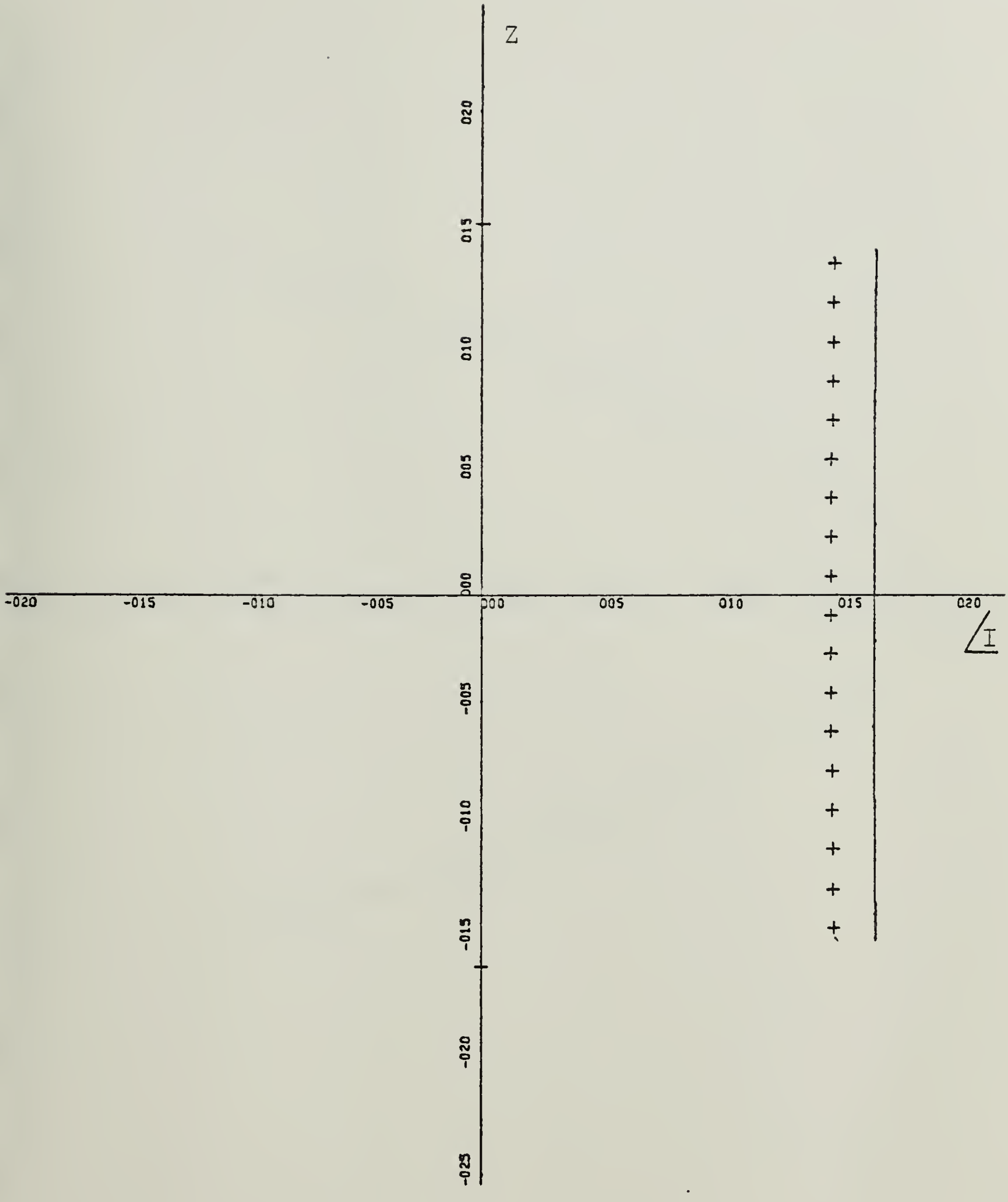
X-SCALE=5.00E-04 UNITS INCH.
Y-SCALE=5.00E-01 UNITS INCH.
CURRENT MAGNITUDE :FEM/T=SOLID :INT.EQ=X
ZL=.5 : AL=1/50 : DR=DZ/2 : ALFA=90



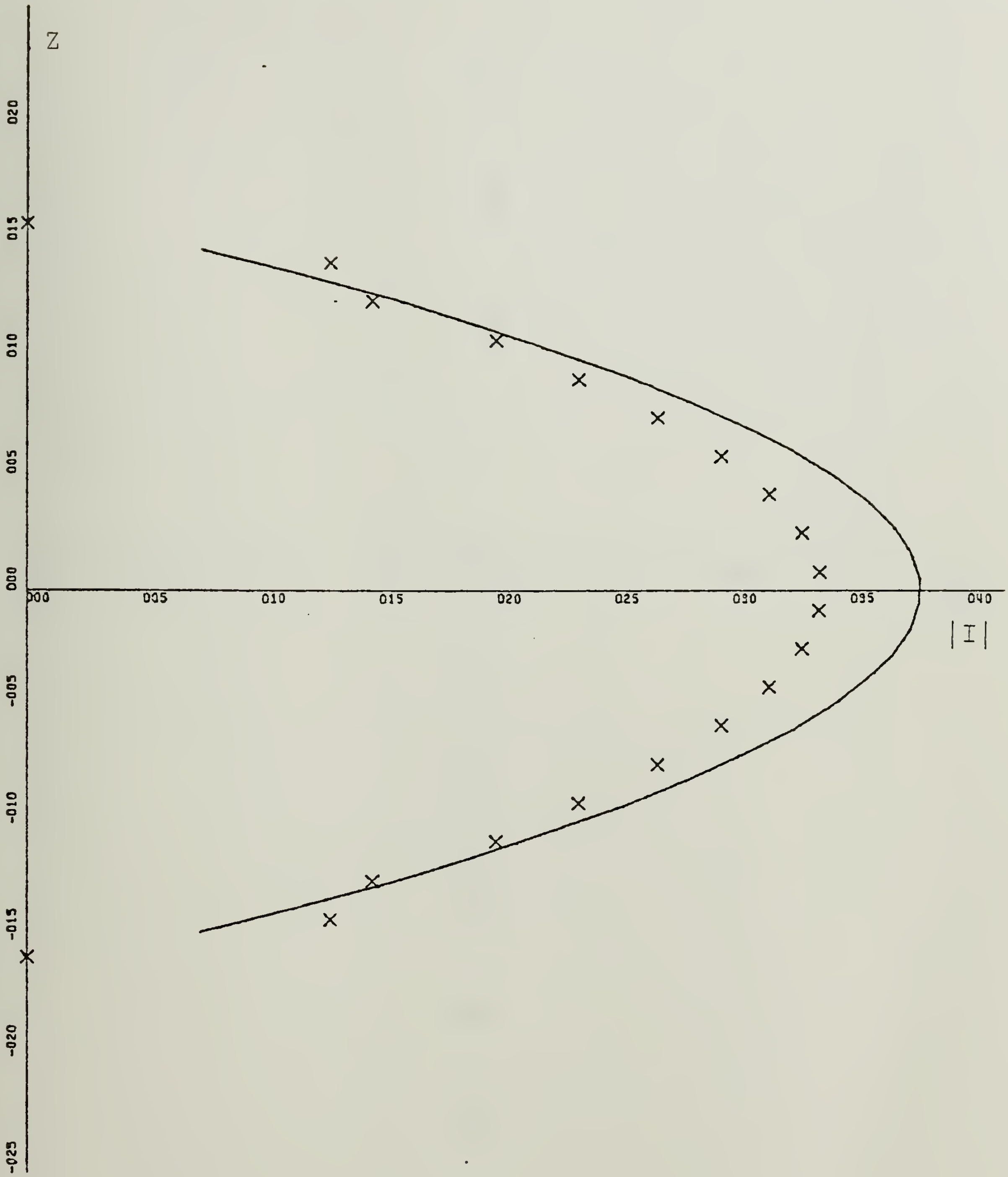
X-SCALE=5.00E+01 UNITS INCH.
Y-SCALE=5.00E-01 UNITS INCH.
CURRENT PHASE :FEM/T=SOLID :INT.EQ=PLUS
ZL=.5 : AL=1/50 : DR=DZ/2 : ALFA=90



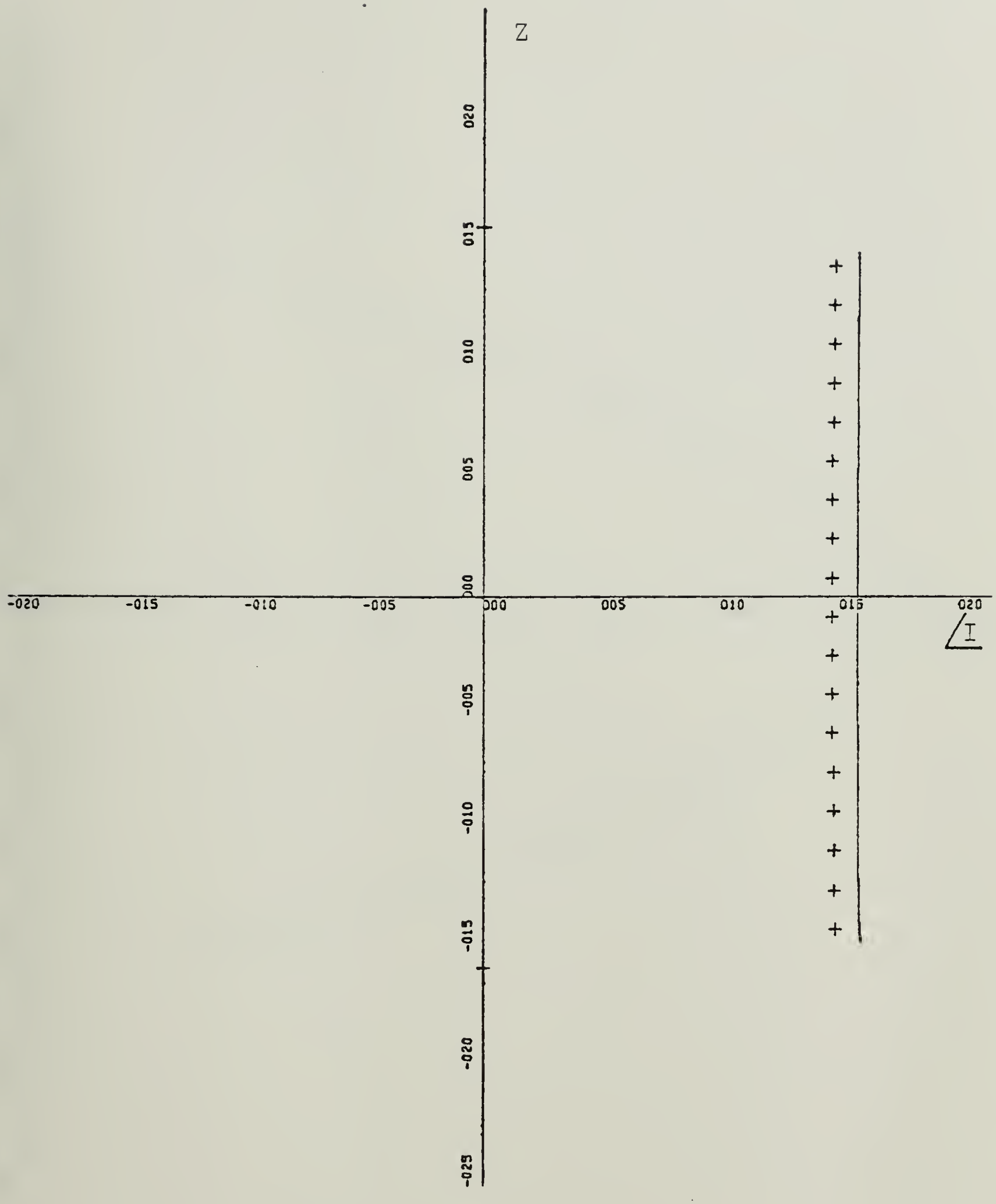
X-SCALE=5.00E-04 UNITS INCH.
Y-SCALE=5.00E-01 UNITS INCH.
CURRENT MAGNITUDE :FEM/T=SOLID :INT.EQ=X
ZL=.5 : AL=1/50 : DR=DZ/4 : ALFA=90



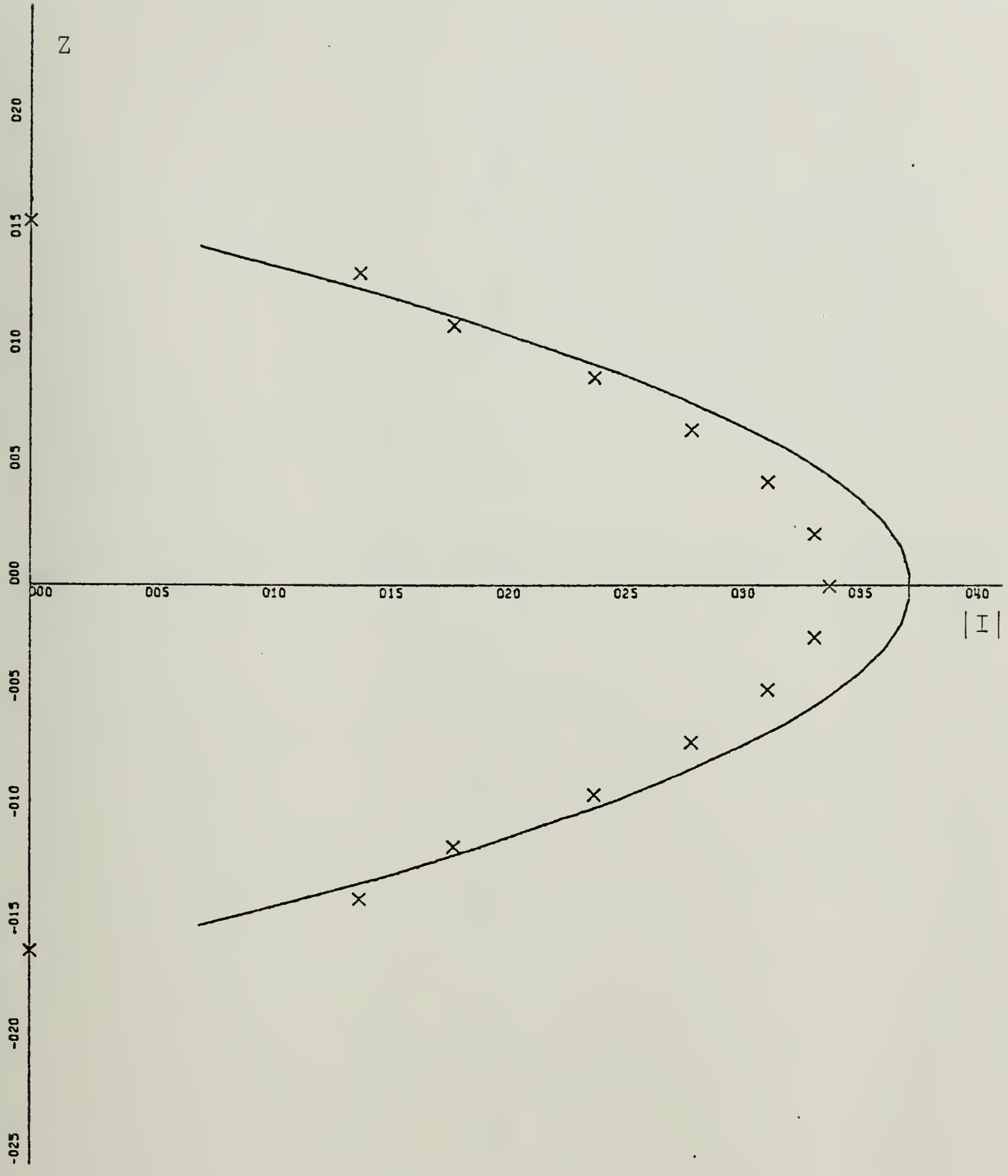
X-SCALE=5.00E+01 UNITS INCH.
Y-SCALE=5.00E-01 UNITS INCH.
CURRENT PHASE :FEM/T=SOLID :INT.EQ=PLUS
ZL=.5 : AL=1/50 : DR=DZ/4 : ALFA=90



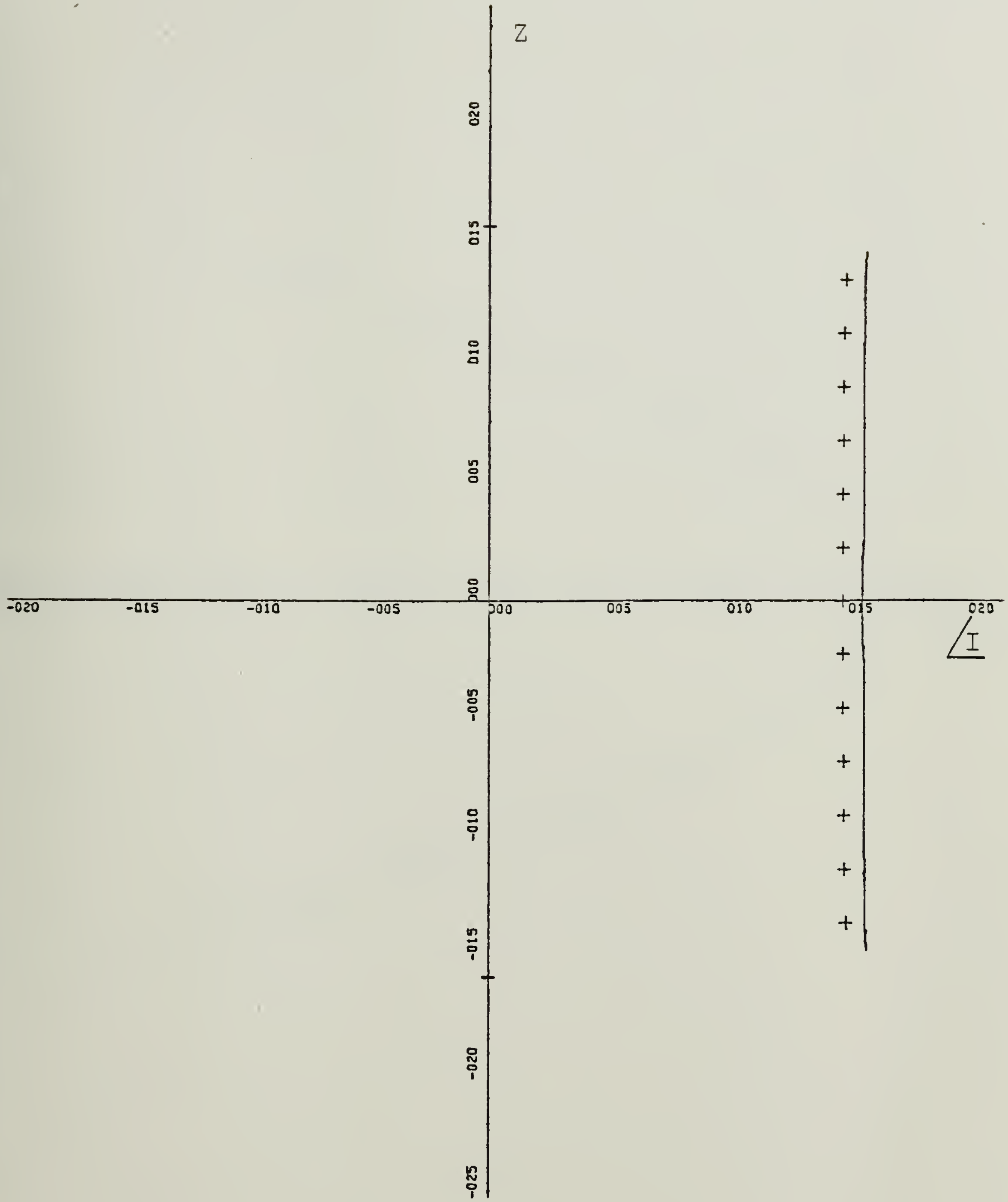
X-SCALE=5.00E-04 UNITS INCH.
Y-SCALE=5.00E-01 UNITS INCH.
CURRENT MAGNITUDE :FEM/T=SOLID :INT.EQ=X
ZL=.5 : AL=1/50 : DR=DZ/8 : ALFA=90



X-SCALE=5.00E+01 UNITS INCH.
Y-SCALE=5.00E-01 UNITS INCH.
CURRENT PHASE :FEM/T=SOLID :INT.EQ=PLUS
ZL=.5 : AL=1/50 : DR=DZ/8 : ALFA=90

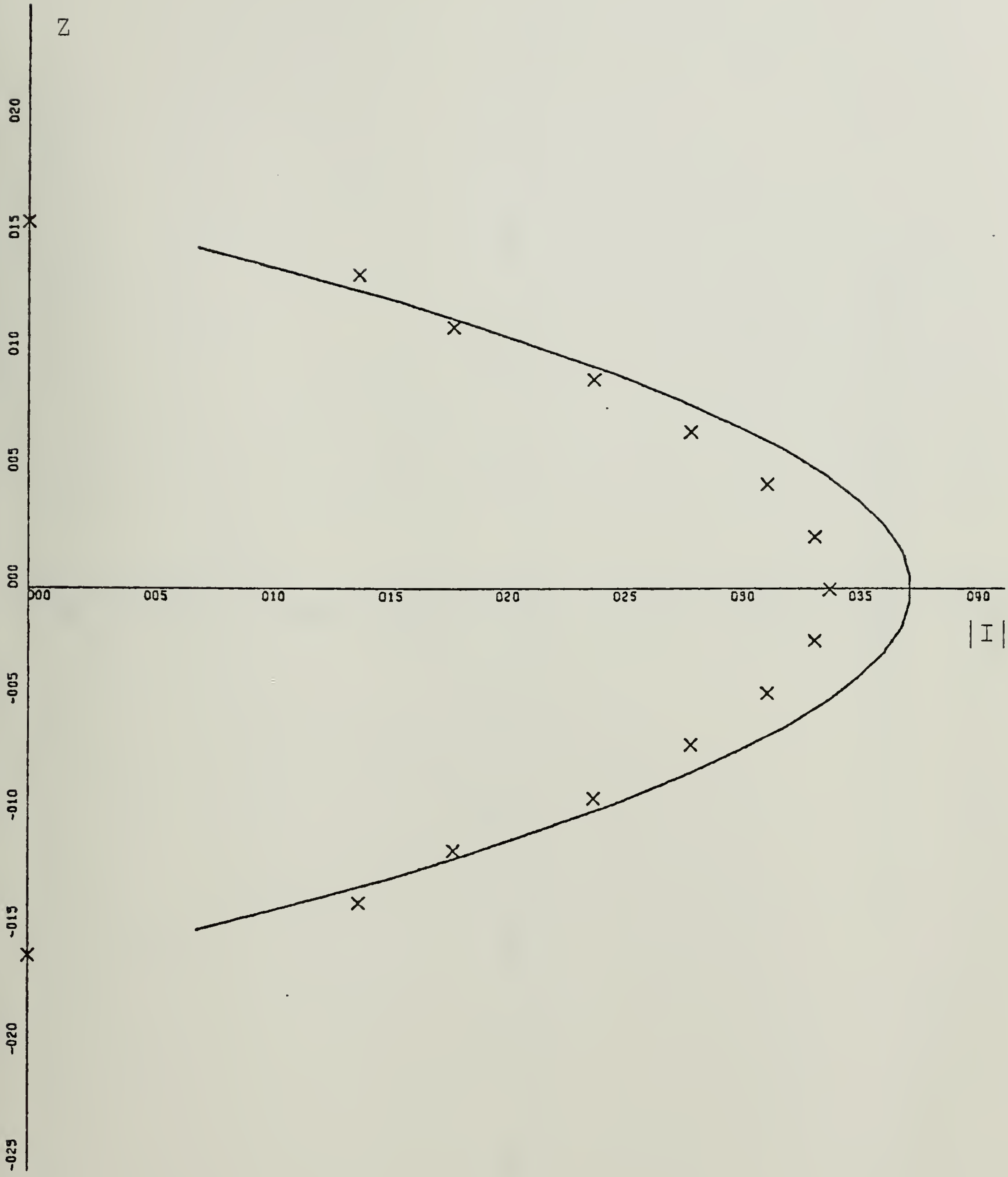


X-SCALE=5.00E-04 UNITS INCH.
Y-SCALE=5.00E-01 UNITS INCH.
CURRENT MAGNITUDE :FEM/T=SOLID :INT.EQ=X
ZL=.5 : AL=1/50 : DR=DZ/32 : ALFA=90



X-SCALE=5.00E+01 UNITS INCH.
Y-SCALE=5.00E-01 UNITS INCH.
CURRENT PHASE :FEM/T=SOLID :INT.EQ=PLUS
ZL=.5 : AL=1/50 : DR=DZ/32 : ALFA=90

A-36



X-SCALE=5.00E-04 UNITS INCH.
Y-SCALE=5.00E-01 UNITS INCH.
CURRENT MAGNITUDE :FEM/T=SOLID :INT.EQ=X
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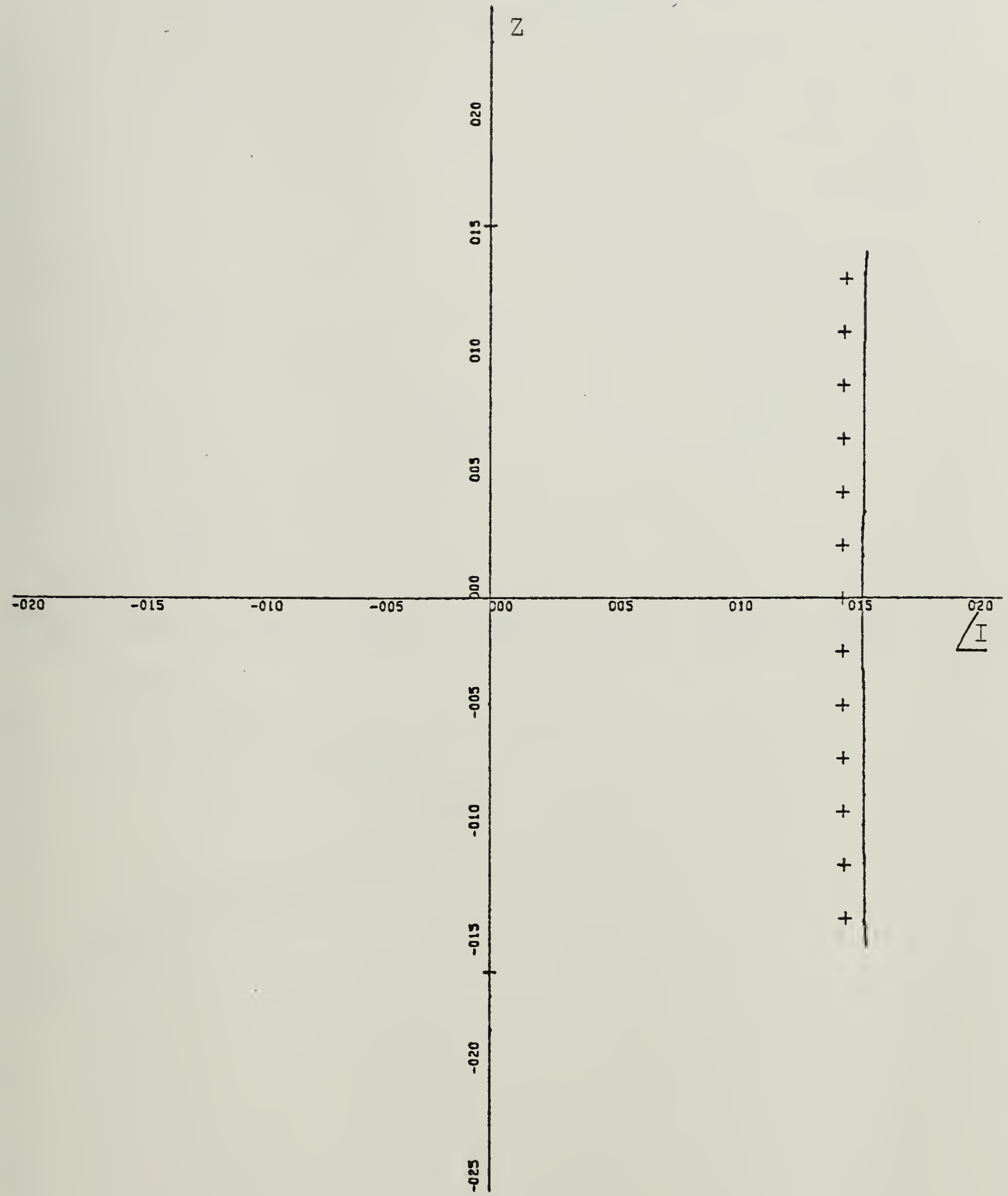
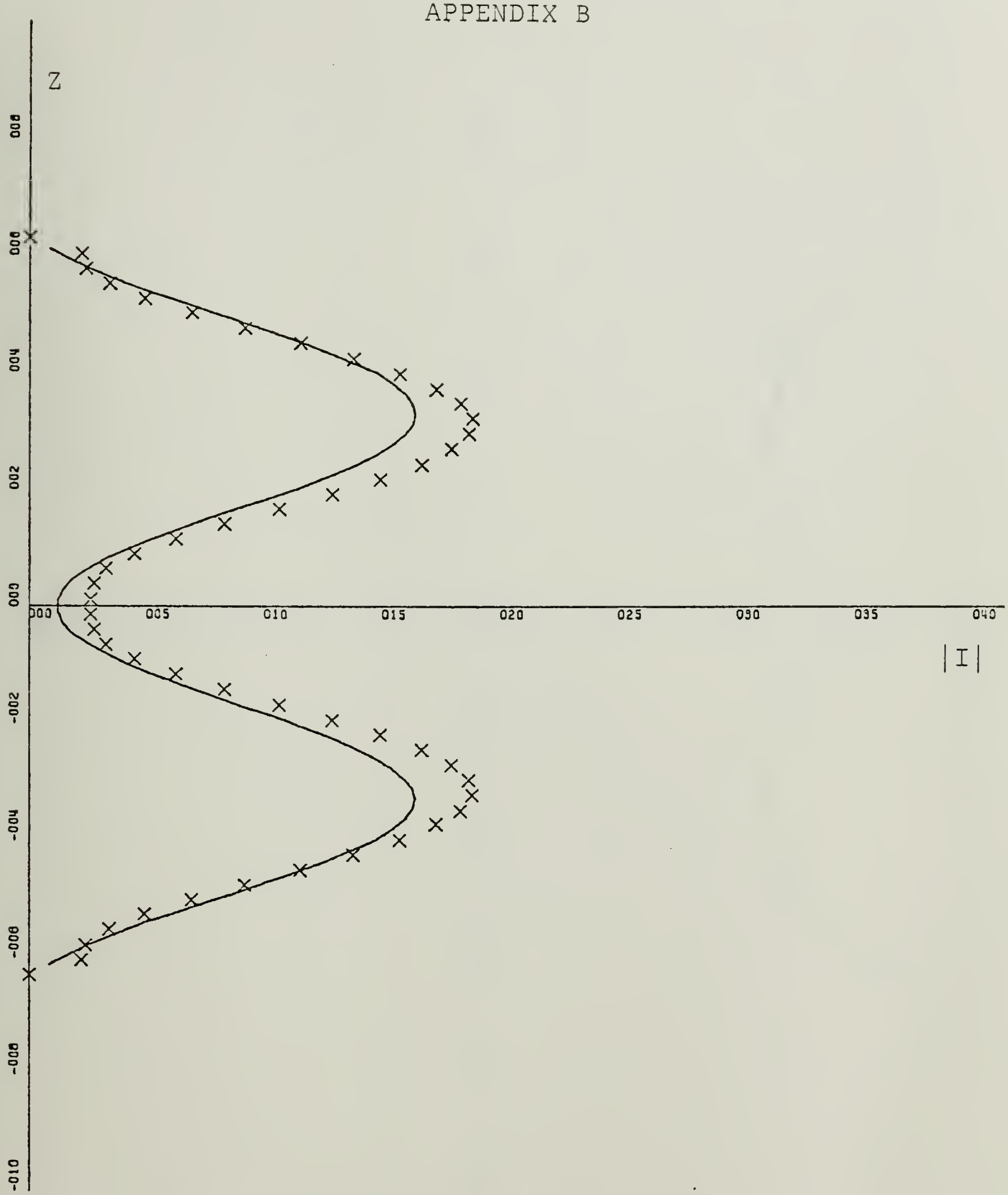


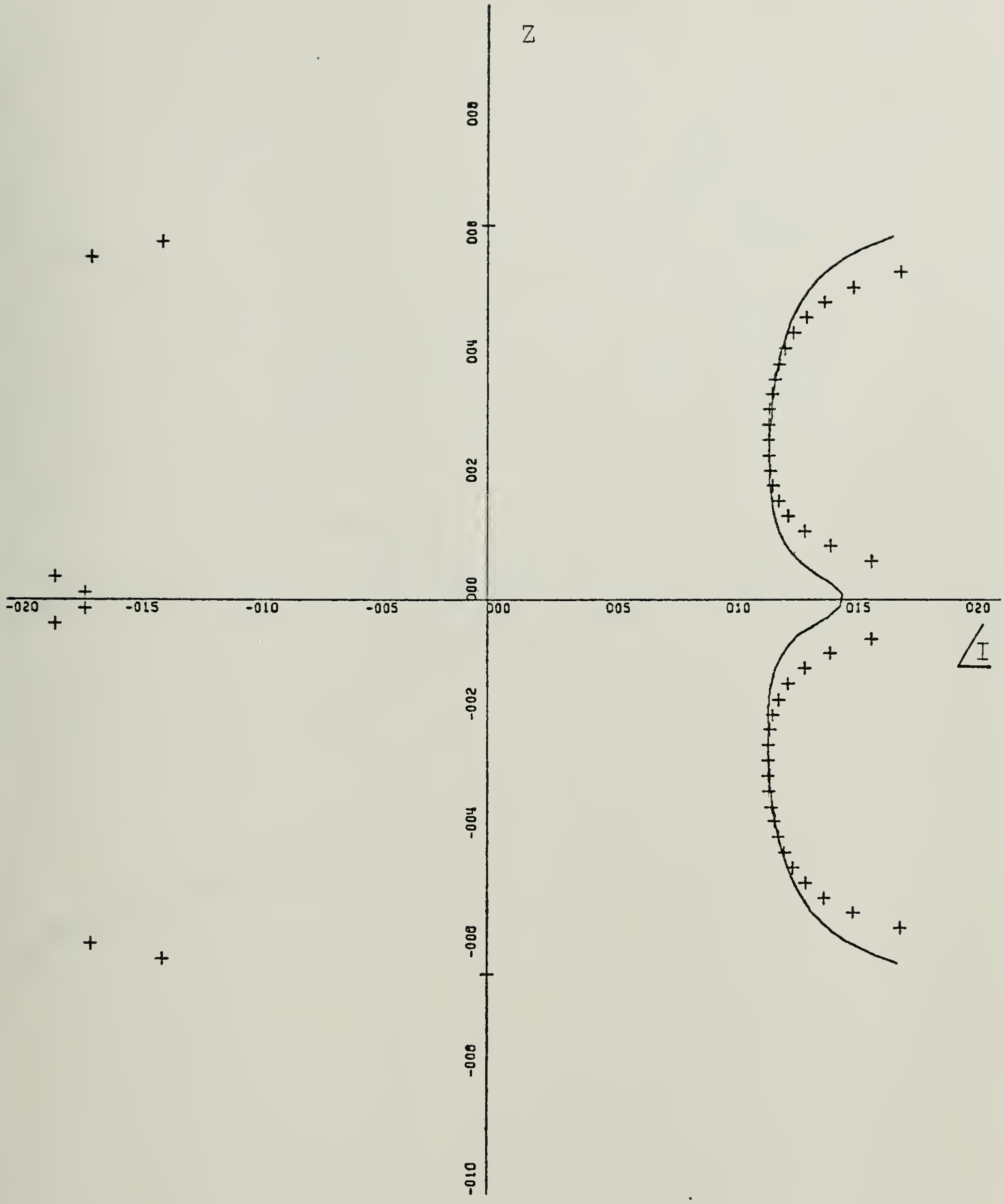
X-SCALE=5.00E+01 UNITS INCH.
Y-SCALE=5.00E-01 UNITS INCH.
CURRENT PHASE :FEM/T=SOLID :INT.EQ=PLUS
ZL=.5 : AL=1/50 : DR=DZ/64 : ALFA=90

A-38

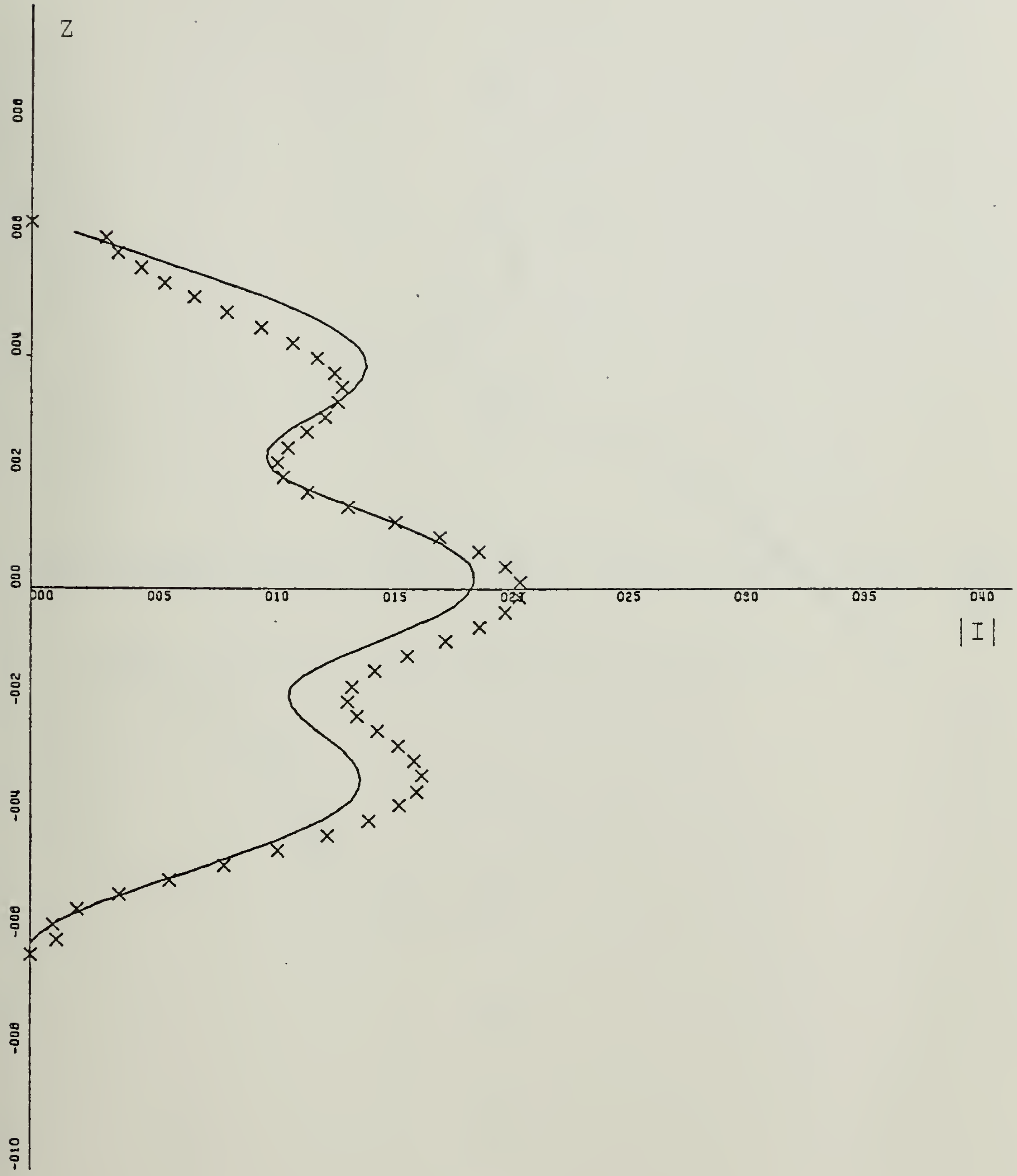
APPENDIX B



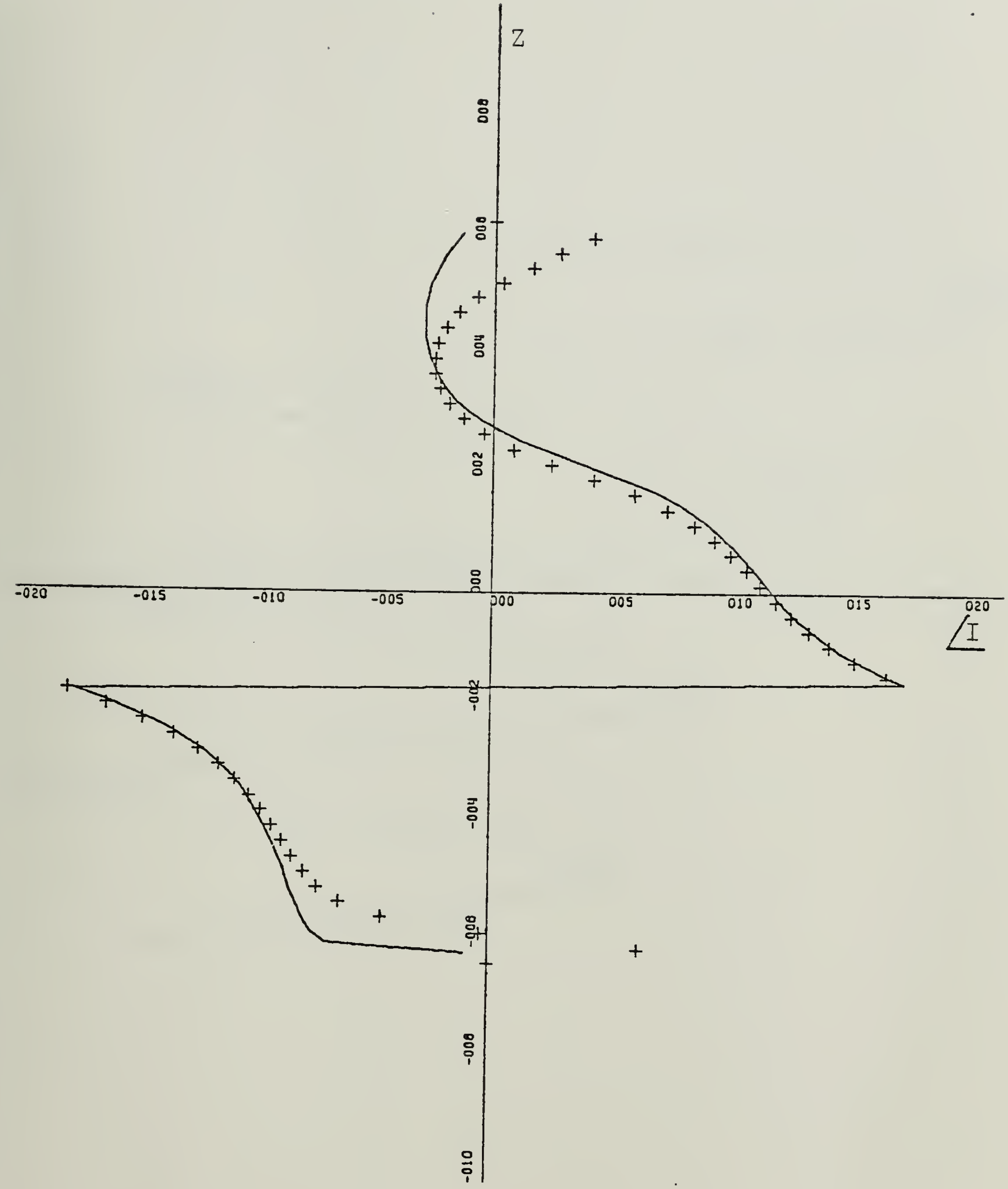
X-SCALE=5.00E-04 UNITS INCH.
Y-SCALE=2.00E+00 UNITS INCH.
CURRENT MAGNITUDE :FEM/T=SOLID :INT.EQ=X
ZL=2.0:AL=0.02:BL=0.001:ALFA=90:NW=70



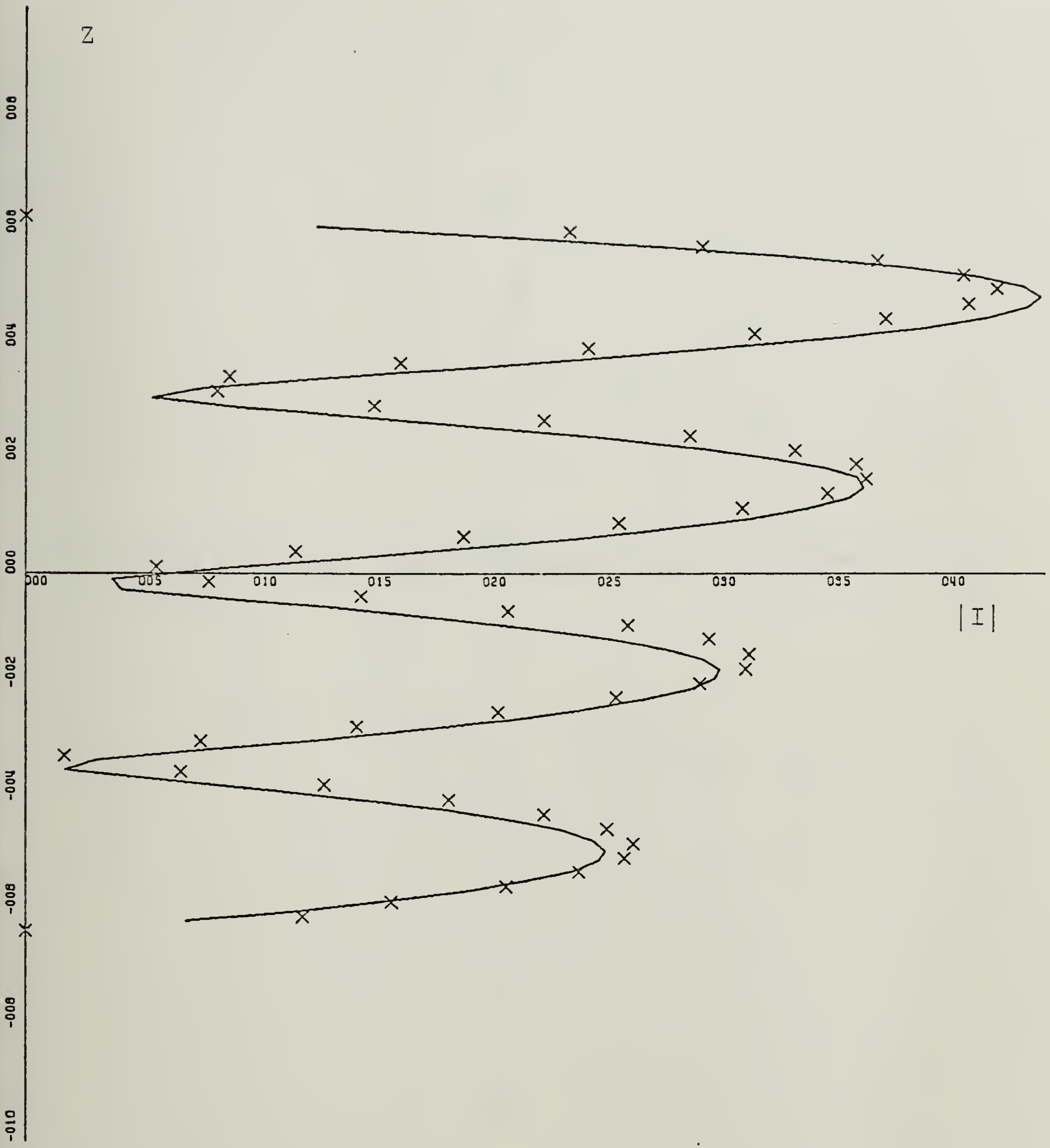
X-SCALE=5.00E+01 UNITS INCH.
Y-SCALE=2.00E+00 UNITS INCH.
CURRENT PHASE :FEM/T=SOLID :INT.EQ=PLUS
ZL=2.0:AL=0.02:BL=0.001:ALFA=90:NW=70



X-SCALE=5.00E-04 UNITS INCH.
Y-SCALE=2.00E+00 UNITS INCH.
CURRENT MAGNITUDE :FEM/T=SOLID :INT.EQ=X
ZL=2.0:AL=0.02:BL=0.001:ALFA=60:NW=70



X-SCALE=5.00E+01 UNITS INCH.
Y-SCALE=2.00E+00 UNITS INCH.
CURRENT PHASE :FEM/T=SOLID :INT.EQ=PLUS
ZL=2.0:AL=0.02:BL=0.001:ALFA=60:NW=70



X-SCALE=5.00E-04 UNITS INCH.
Y-SCALE=2.00E+00 UNITS INCH.
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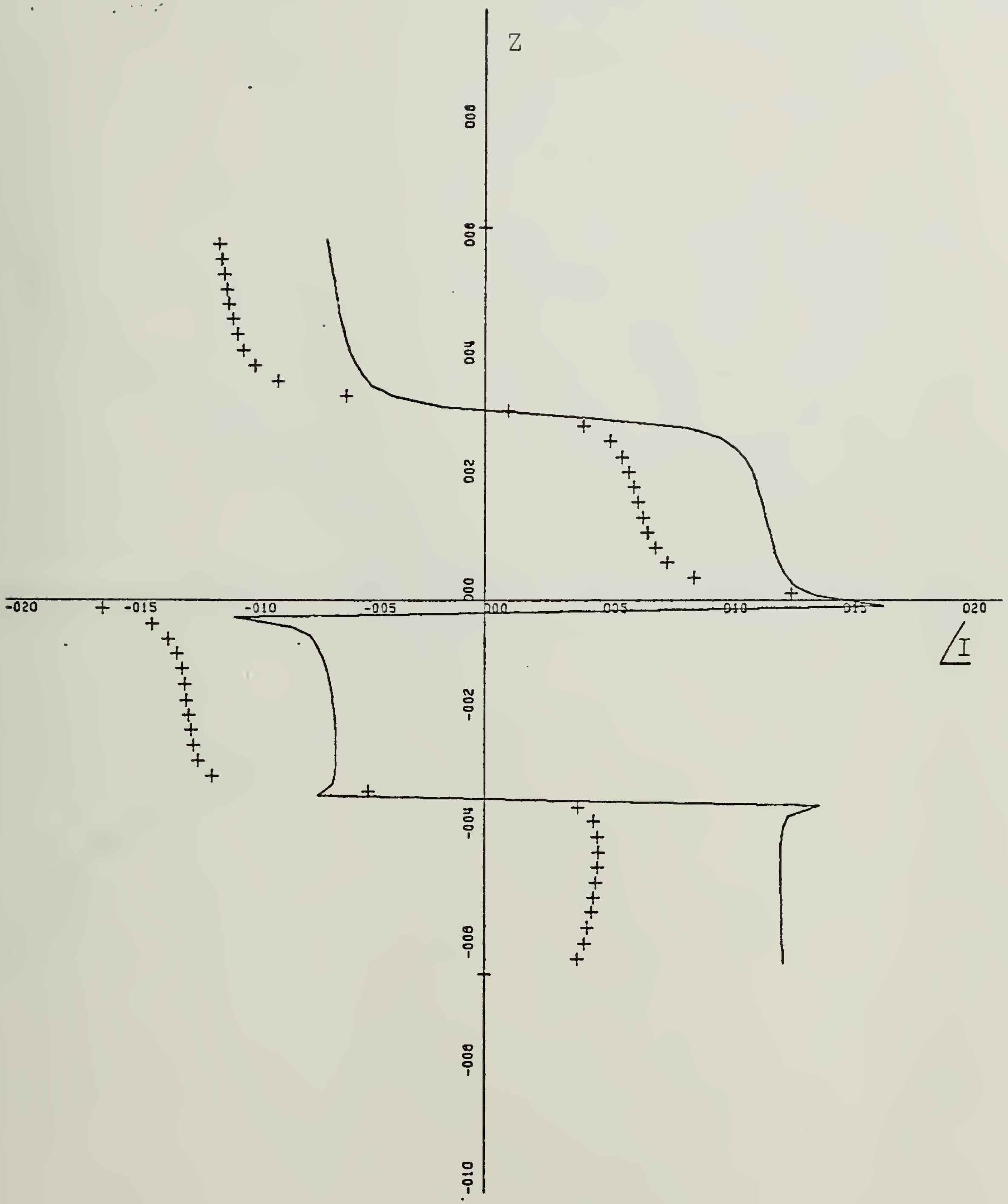
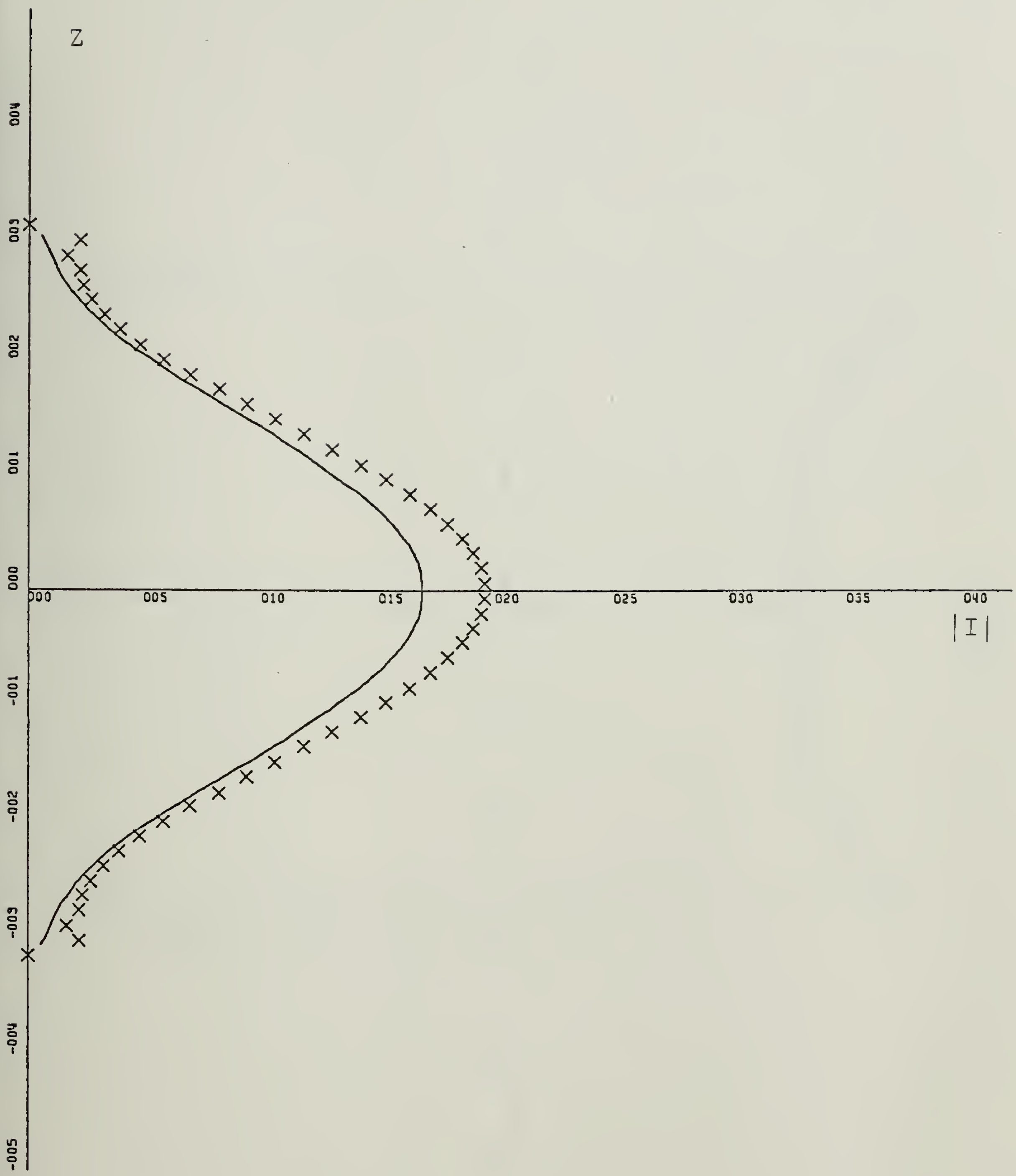
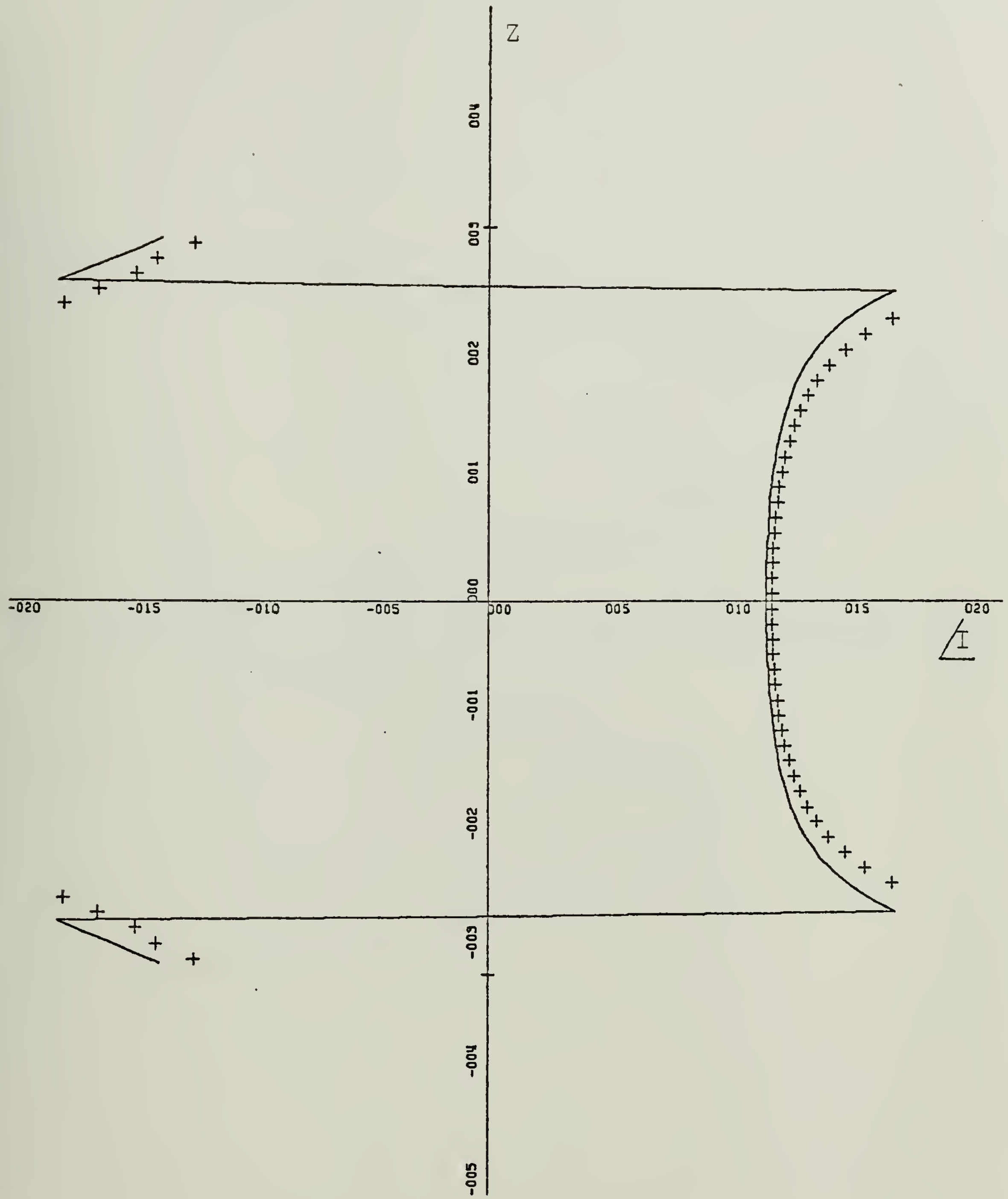
ZL=2.0:AL=0.02:BL=0.001:ALFA=30:NW=70



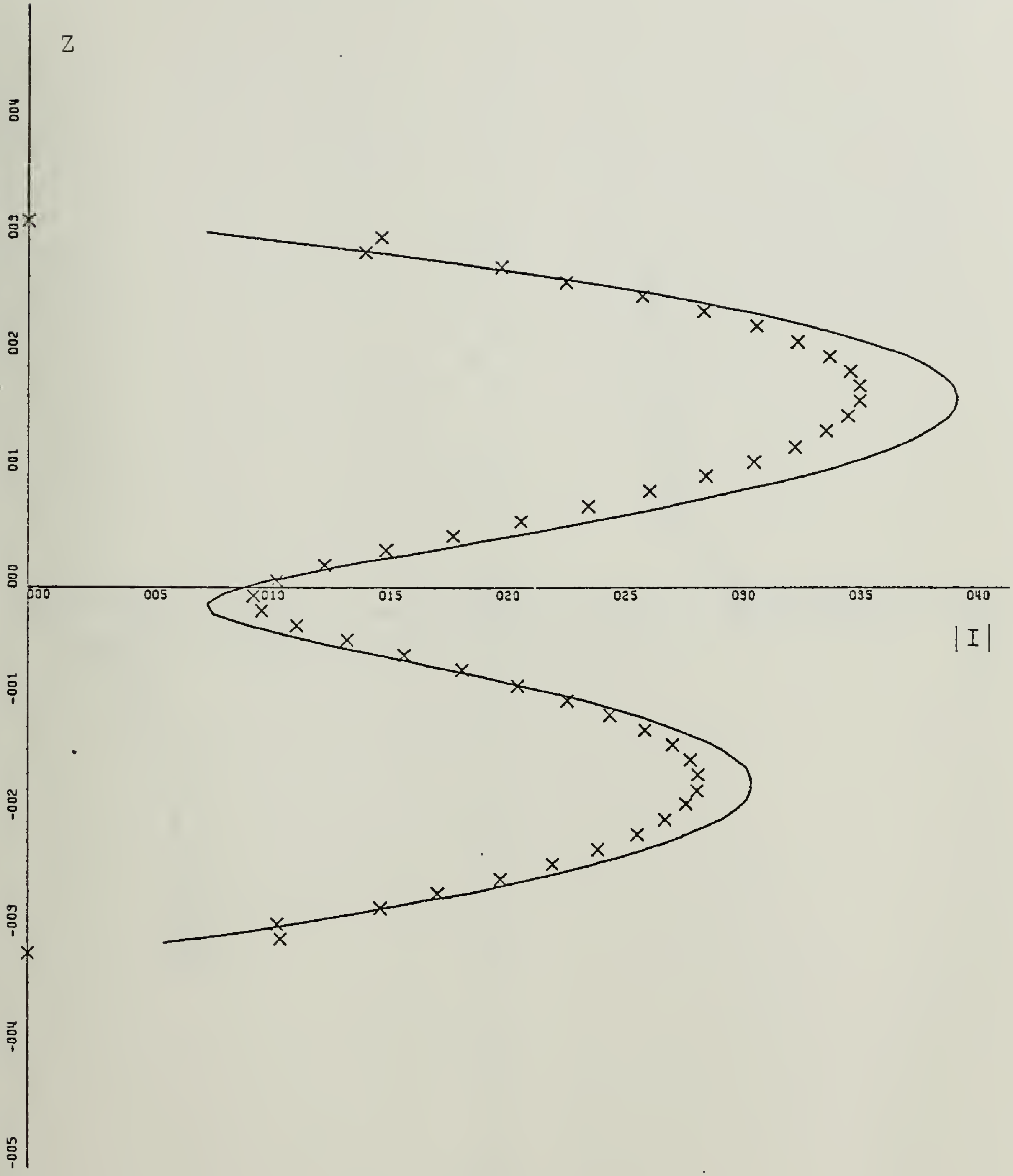
X-SCALE=5.00E+01 UNITS INCH.
Y-SCALE=2.00E+00 UNITS INCH.
CURRENT PHASE :FEM/T=SOLID :INT.EQ=PLUS
ZL=2.0:AL=0.02:BL=0.001:ALFA=30:NW=70



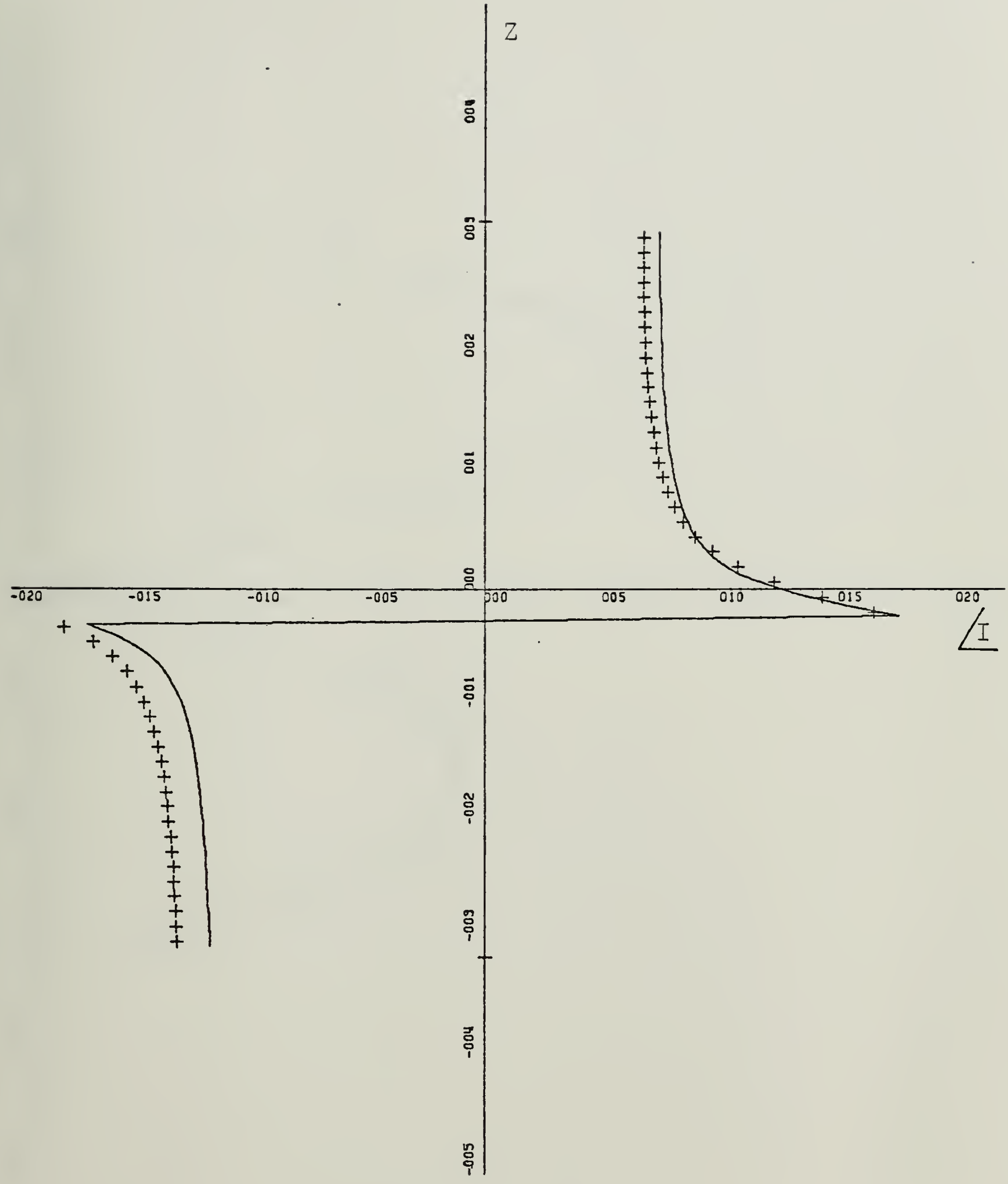
X-SCALE=5.00E-04 UNITS INCH.
Y-SCALE=1.00E+00 UNITS INCH.
CURRENT MAGNITUDE :FEM/T=SOLID :INT.EQ=X
ZL=1.0:AL=0.02:BL=0.001:ALFA=90:NW=70



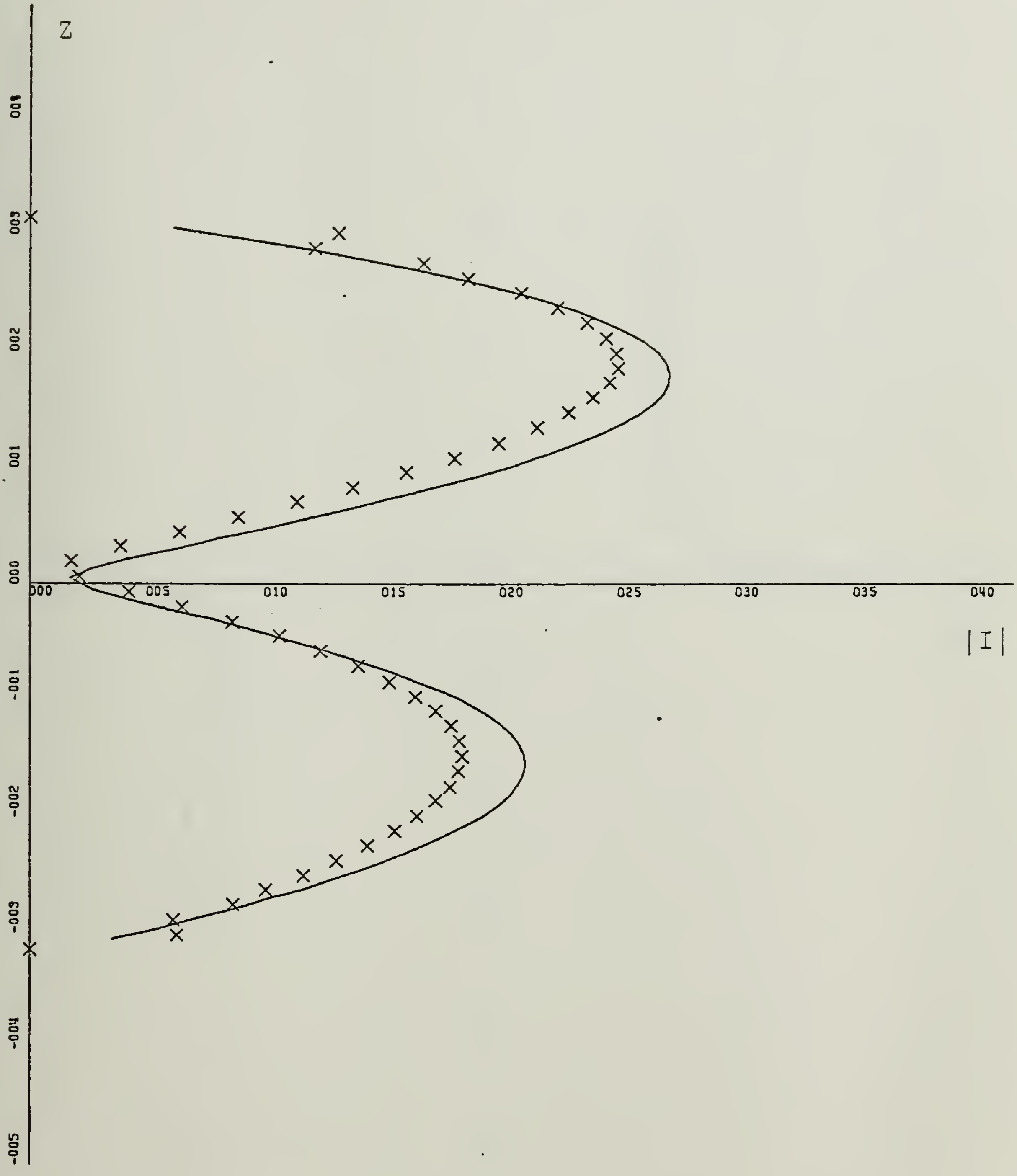
X-SCALE=5.00E+01 UNITS INCH.
Y-SCALE=1.00E+00 UNITS INCH.
CURRENT PHASE :FEM/T=SOLID :INT.EQ=PLUS
ZL=1.0:AL=0.02:BL=0.001:ALFA=90:NW=70



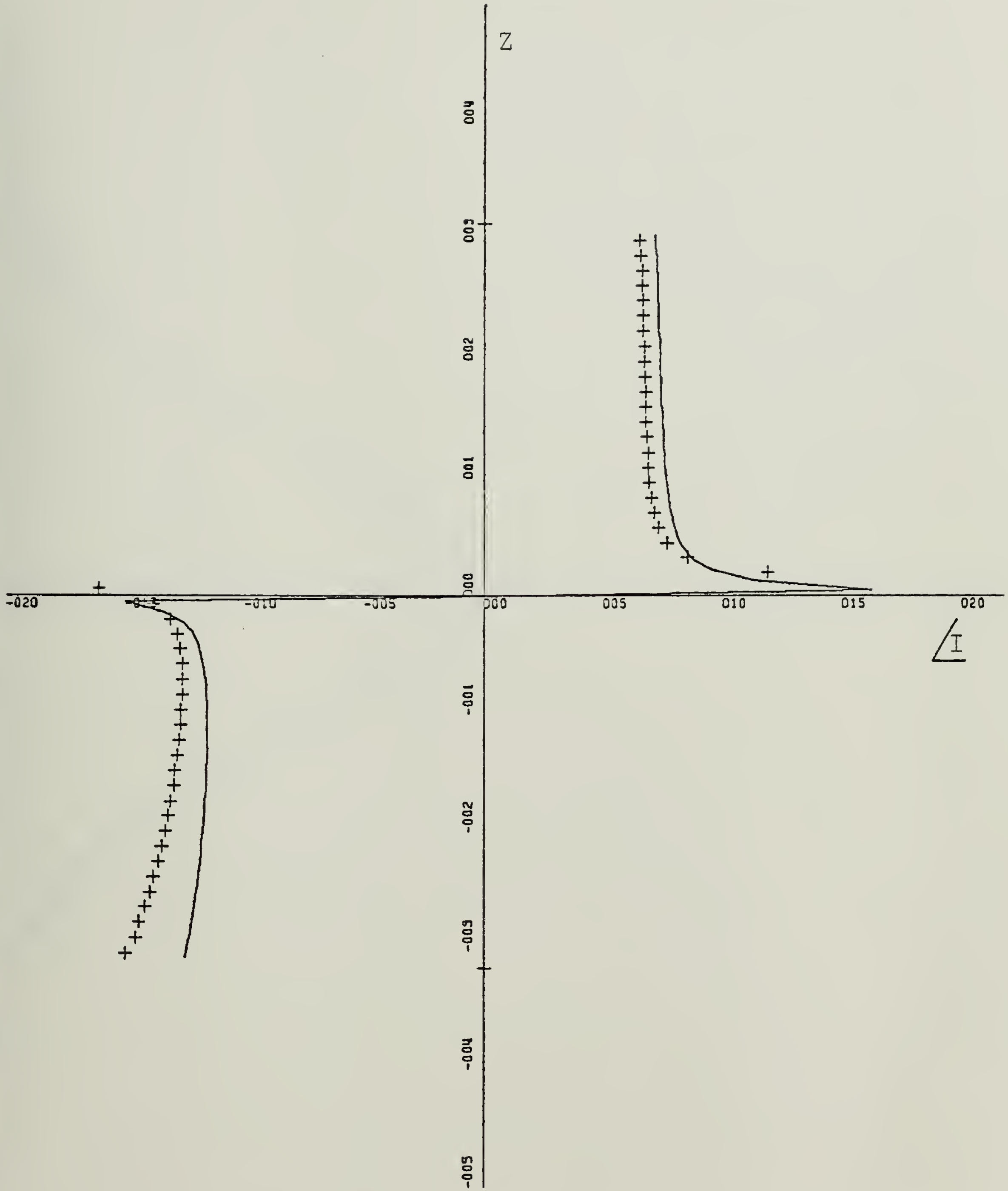
X-SCALE=5.00E-04 UNITS INCH.
Y-SCALE=1.00E+00 UNITS INCH.
CURRENT MAGNITUDE :FEM/T=SOLID :INT.EQ=X
ZL=1.0:AL=0.02:BL=0.001:ALFA=60:NW=70



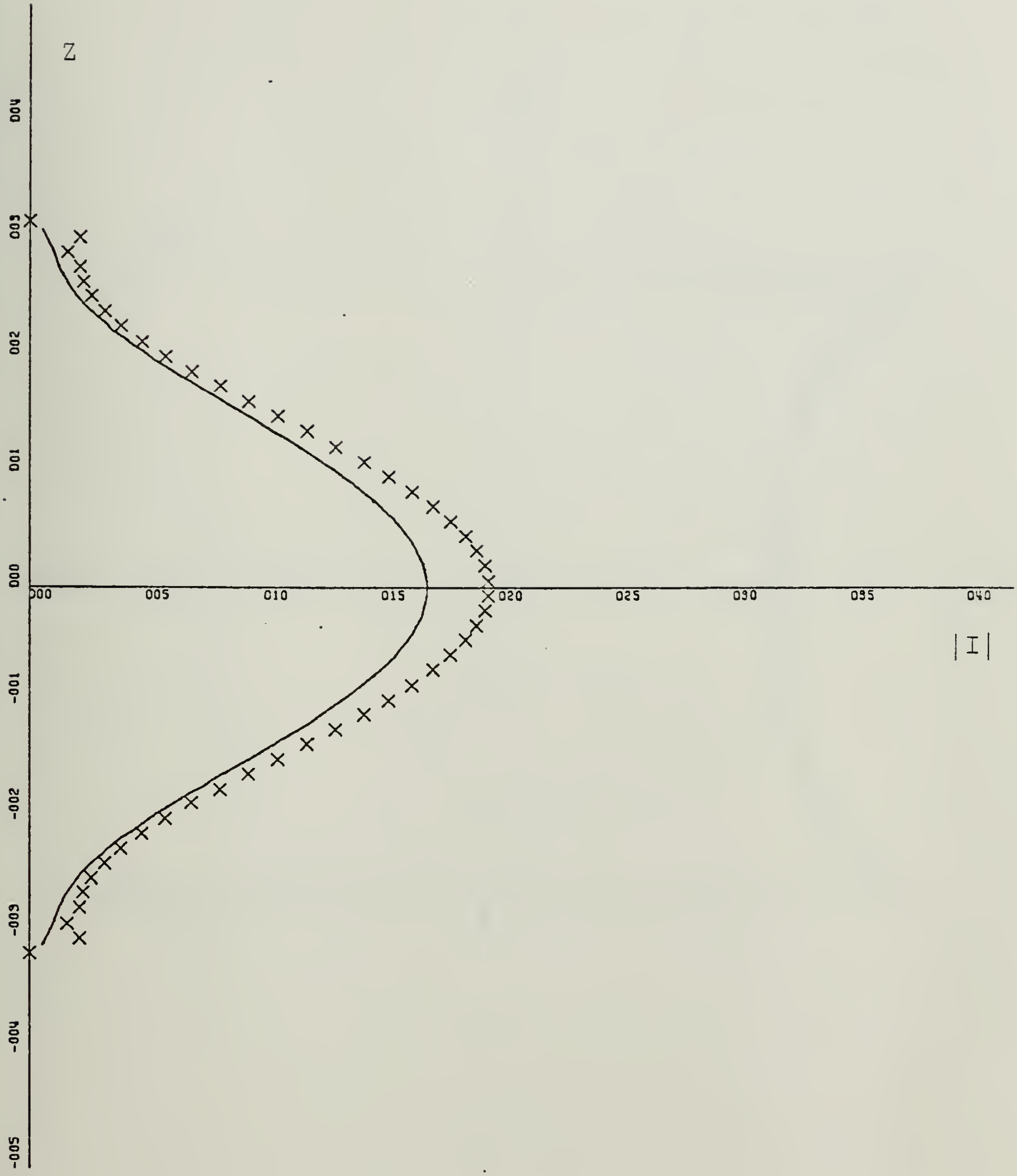
X-SCALE=5.00E+01 UNITS INCH.
Y-SCALE=1.00E+00 UNITS INCH.
CURRENT PHASE :FEM/T=SOLID :INT.EQ=PLUS
ZL=1.0:AL=0.02:BL=0.001:ALFA=60:NW=70



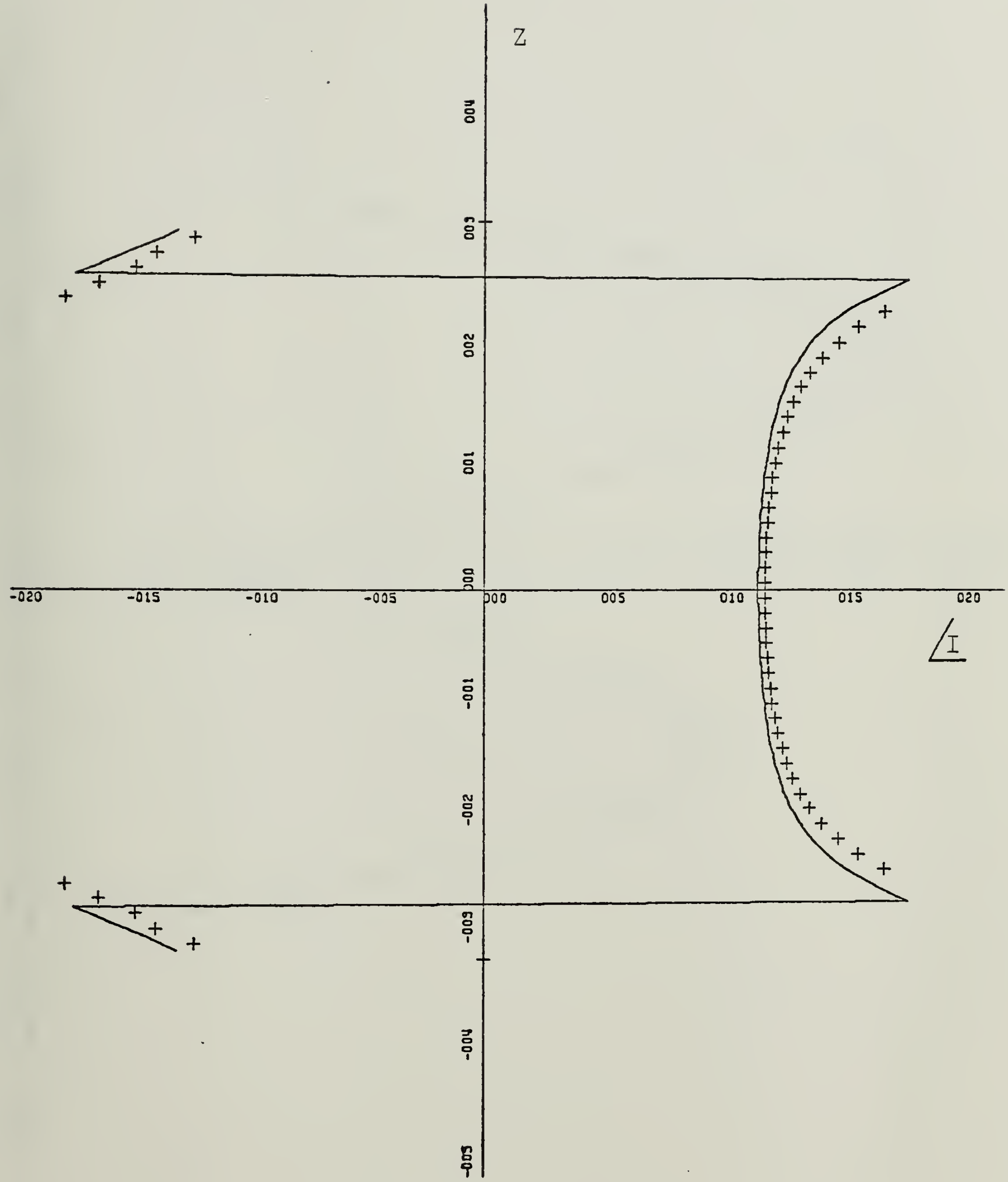
X-SCALE=5.00E-04 UNITS INCH.
Y-SCALE=1.00E+00 UNITS INCH.
CURRENT MAGNITUDE :FEM/T=SOLID :INT.EQ=X
ZL=1.0:AL=0.02:BL=0.001:ALFA=30:NW=70



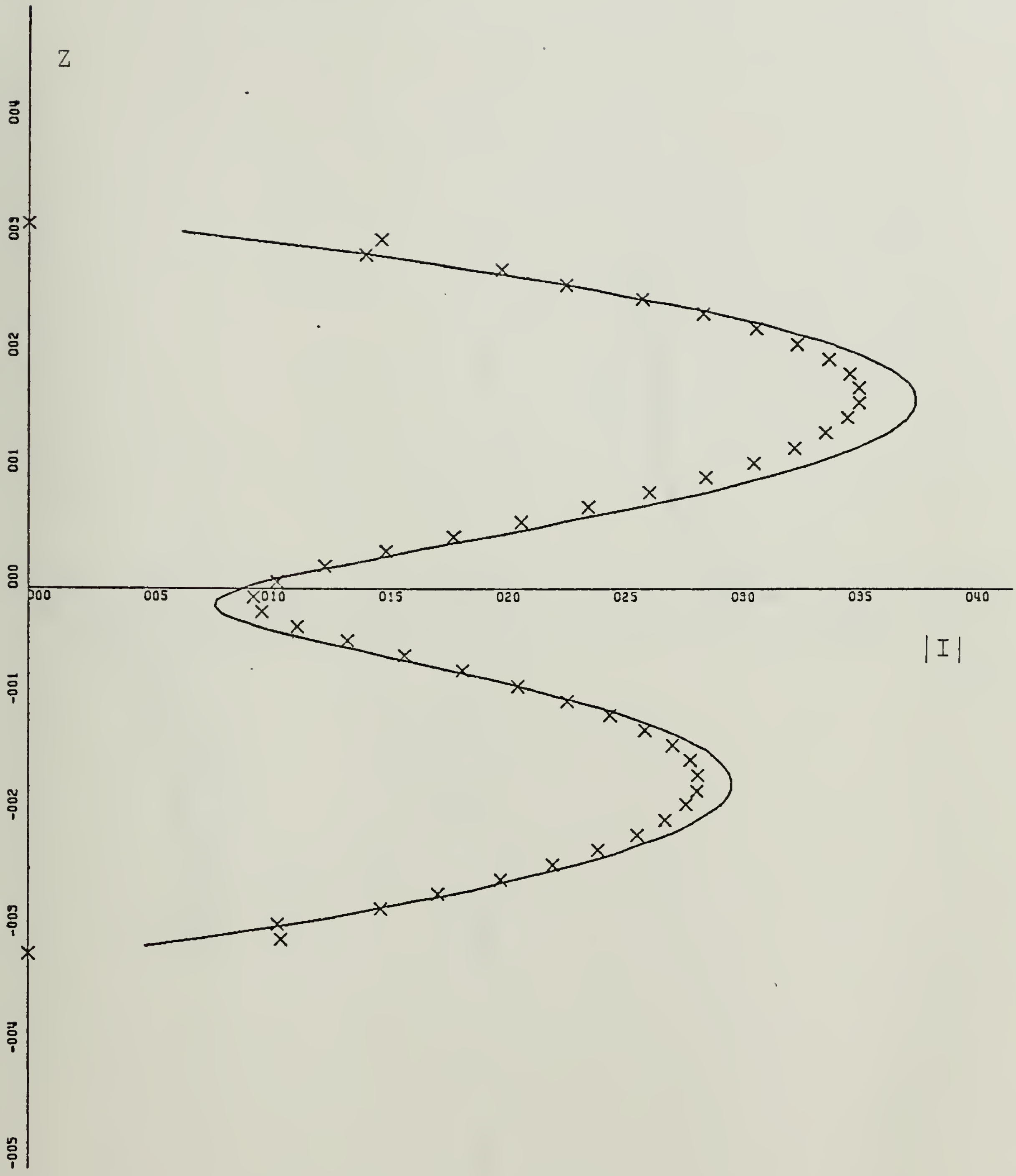
X-SCALE=5.00E+01 UNITS INCH.
Y-SCALE=1.00E+00 UNITS INCH.
CURRENT PHASE :FEM/T=SOLID :INT.EQ=PLUS
ZL=1.0:AL=0.02:BL=0.001:ALFA=30:NW=70



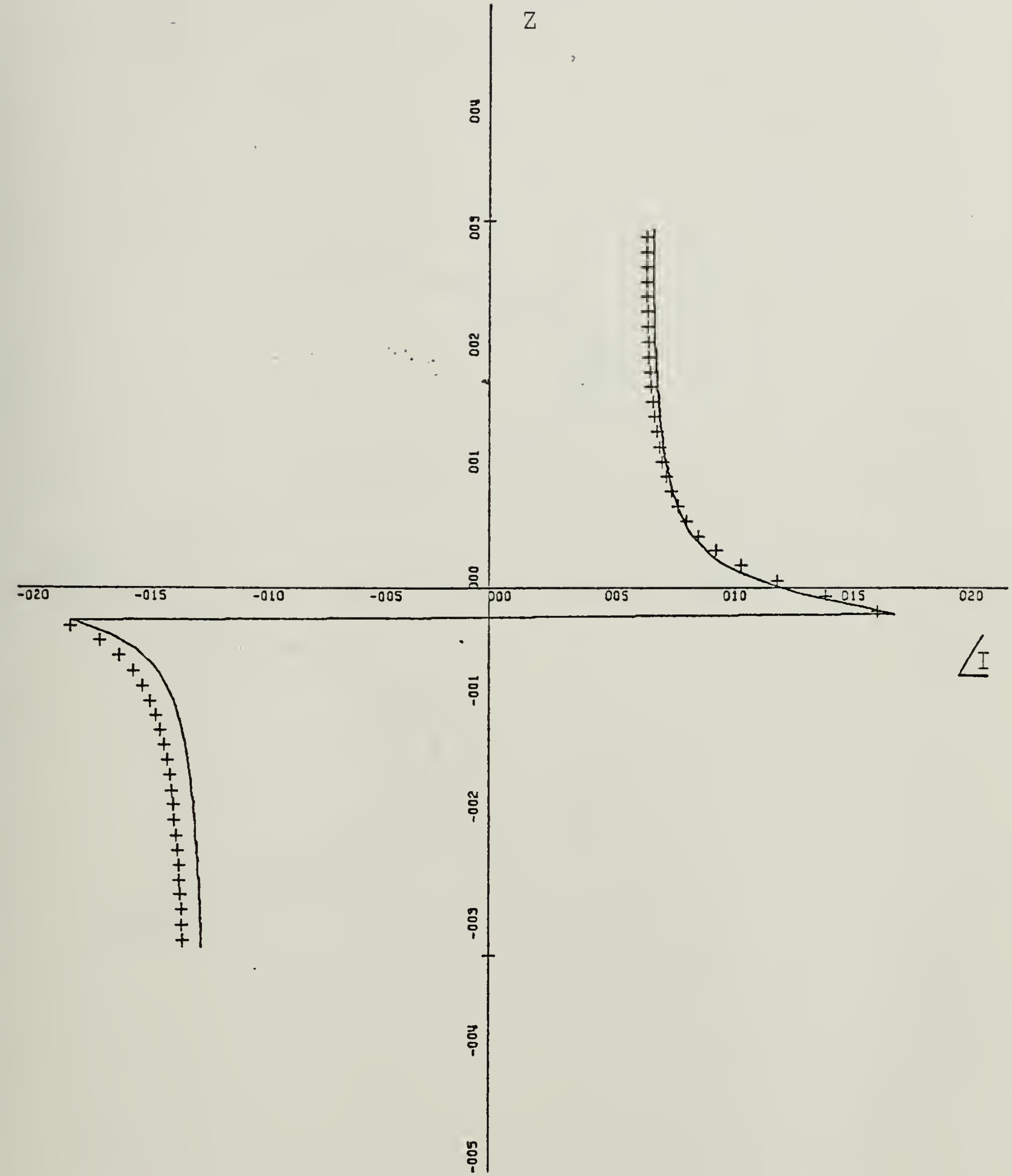
X-SCALE=5.00E-04 UNITS INCH.
Y-SCALE=1.00E+00 UNITS INCH.
CURRENT MAGNITUDE :FEM/T=SOLID :INT.EQ=X
ZL=1.0:AL=0.02:BL=0.001:ALFA=90:NW=100



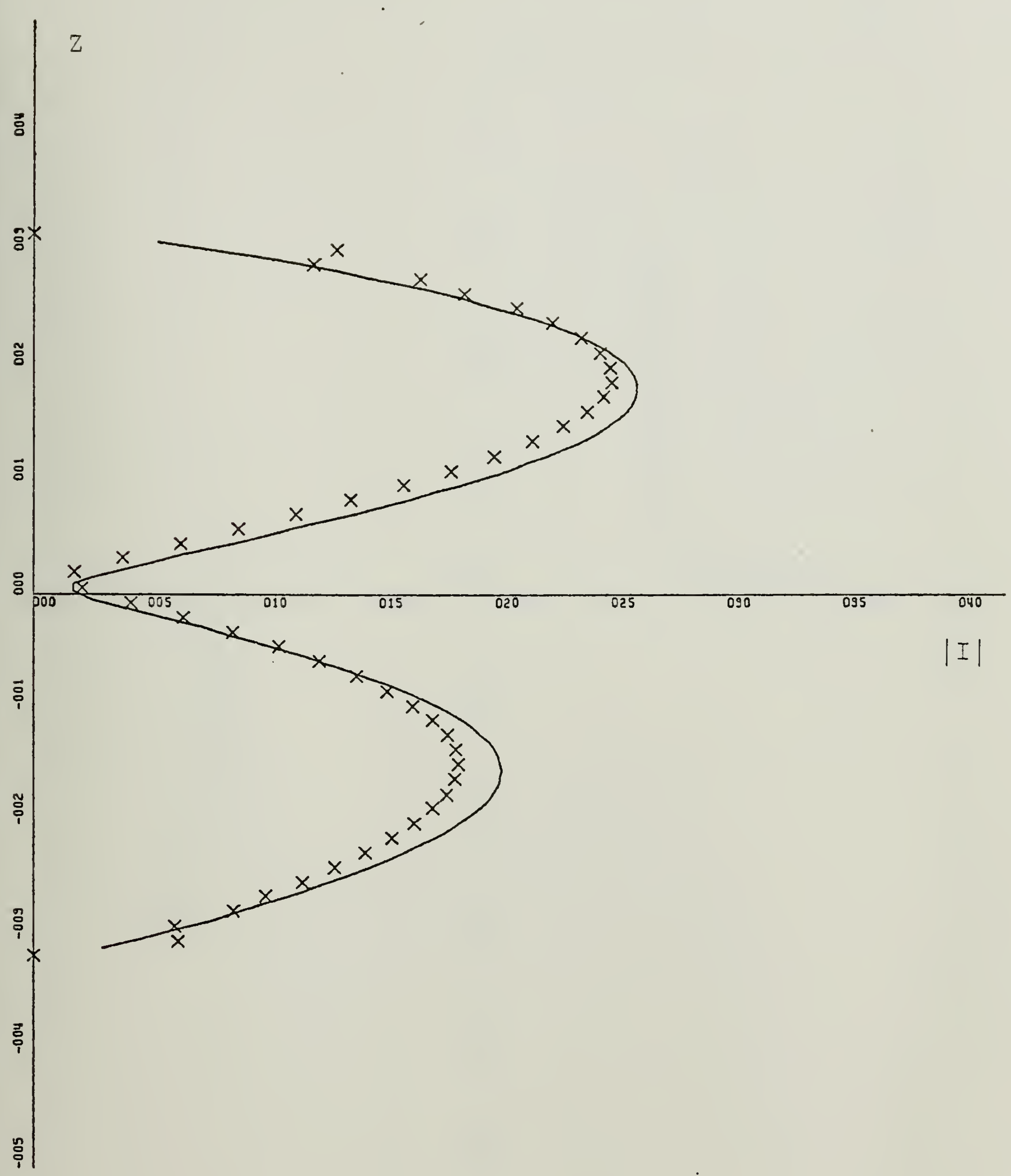
X-SCALE=5.00E+01 UNITS INCH.
Y-SCALE=1.00E+00 UNITS INCH.
CURRENT PHASE :FEM/T=SOLID :INT.EQ=PLUS
ZL=1.0:AL=0.02:BL=0.001:ALFA=90:NW=100



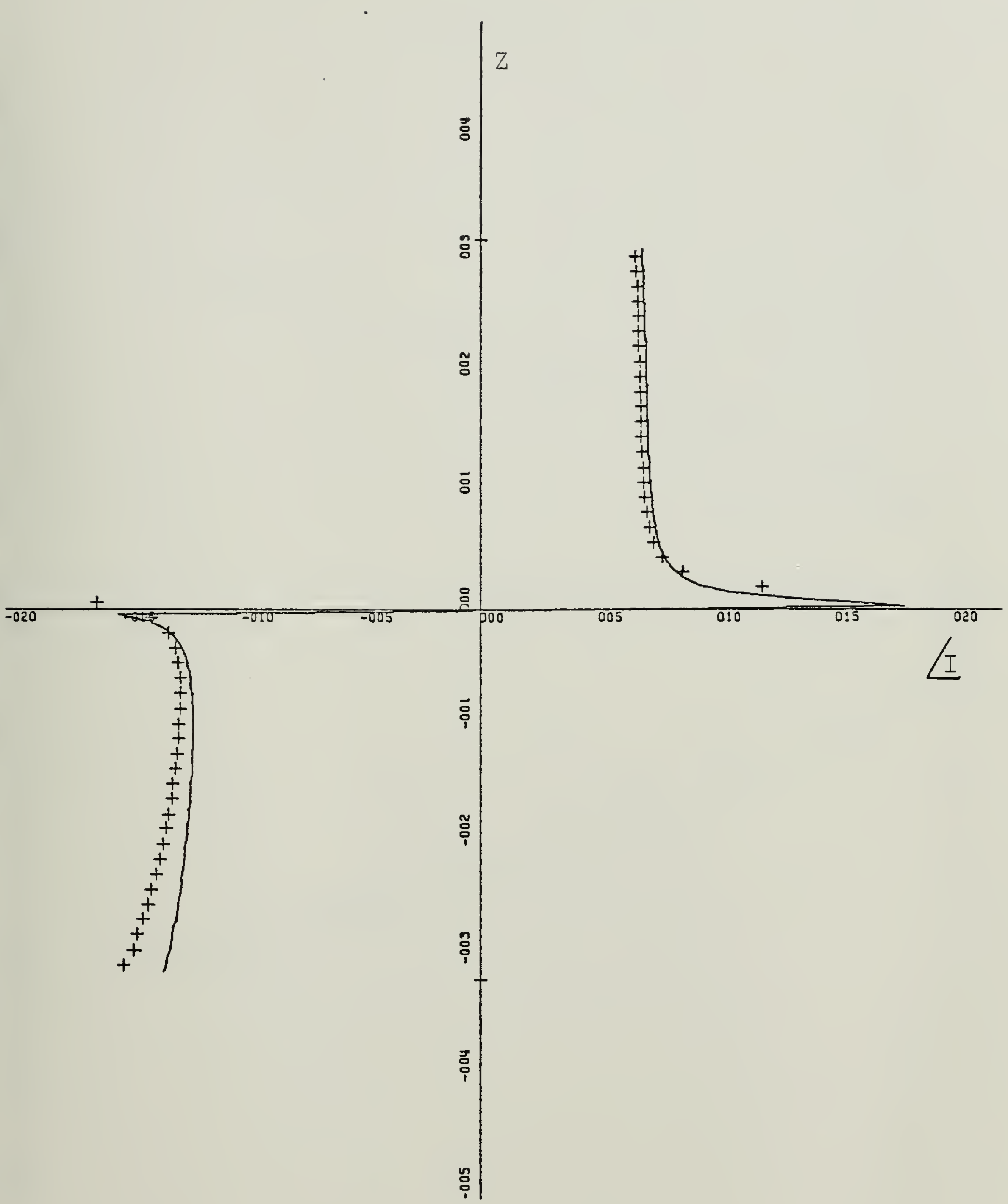
X-SCALE=5.00E-04 UNITS INCH.
Y-SCALE=1.00E+00 UNITS INCH.
CURRENT MAGNITUDE :FEM/T=SOLID :INT.EQ=X
ZL=1.0:AL=0.02:BL=0.001:ALFA=60:NW=100



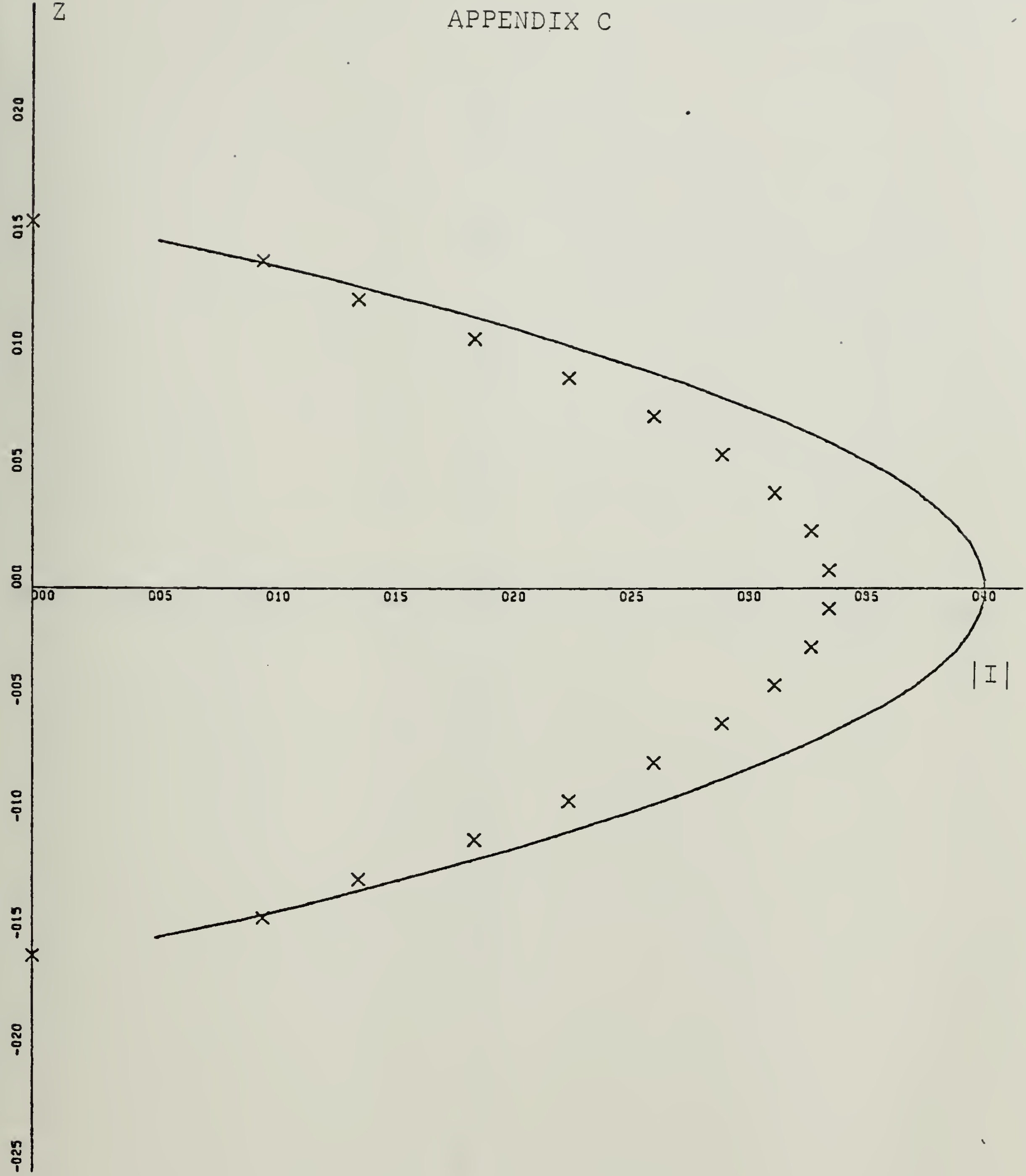
X-SCALE=5.00E+01 UNITS INCH.
Y-SCALE=1.00E+00 UNITS INCH.
CURRENT PHASE :FEM/T=SOLID :INT.EQ=PLUS
ZL=1.0:AL=0.02:BL=0.001:ALFA=60:NW=100



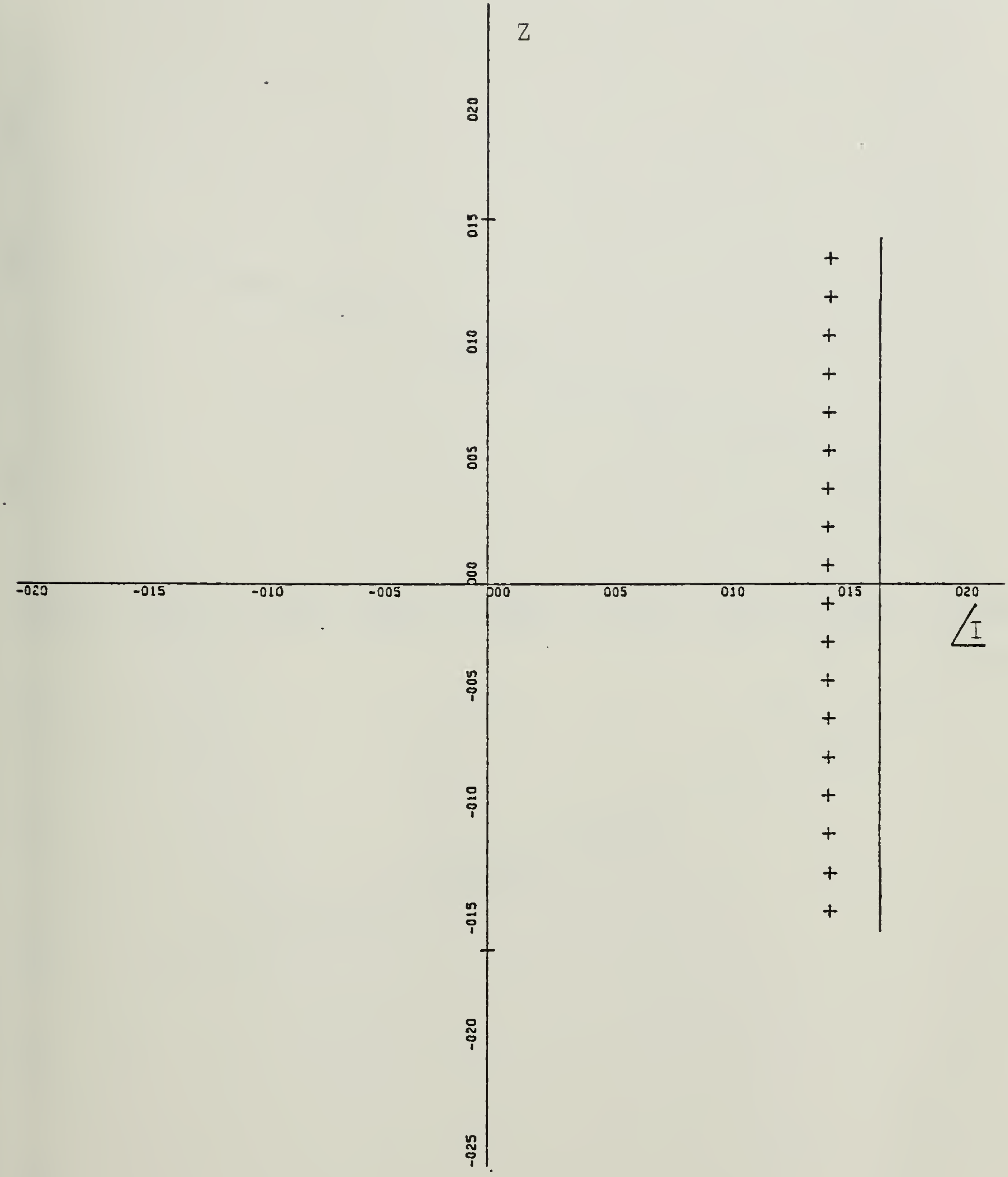
X-SCALE=5.00E-04 UNITS INCH.
Y-SCALE=1.00E+00 UNITS INCH.
CURRENT MAGNITUDE :FEM/T=SOLID :INT.EQ=X
ZL=1.0:AL=0.02:BL=0.001:ALFA=30:NW=100



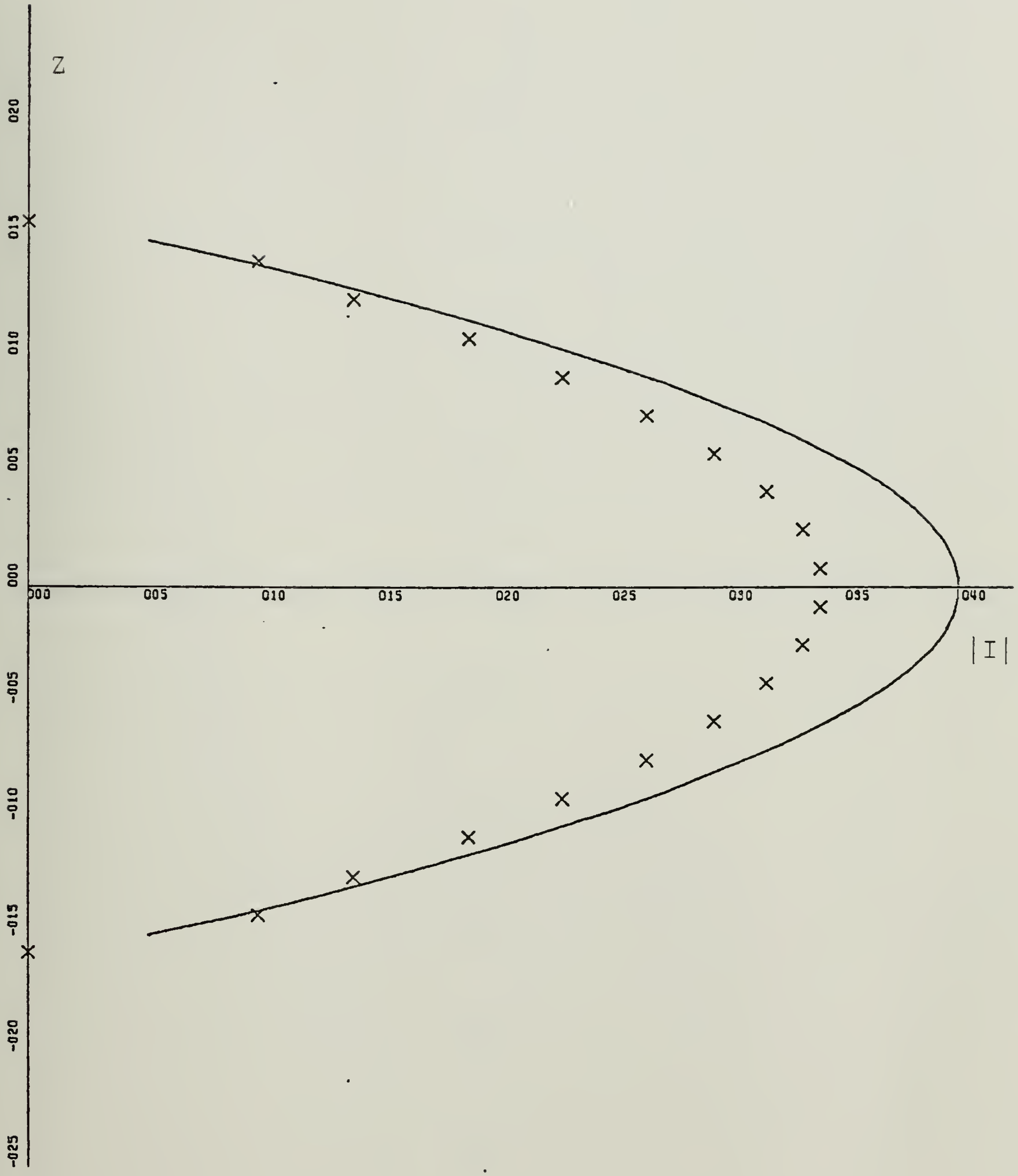
X-SCALE=5.00E+01 UNITS INCH.
Y-SCALE=1.00E+00 UNITS INCH.
CURRENT PHASE :FEM/T=SOLID :INT.EQ=PLUS
ZL=1.0:AL=0.02:BL=0.001:ALFA=30:NW=100



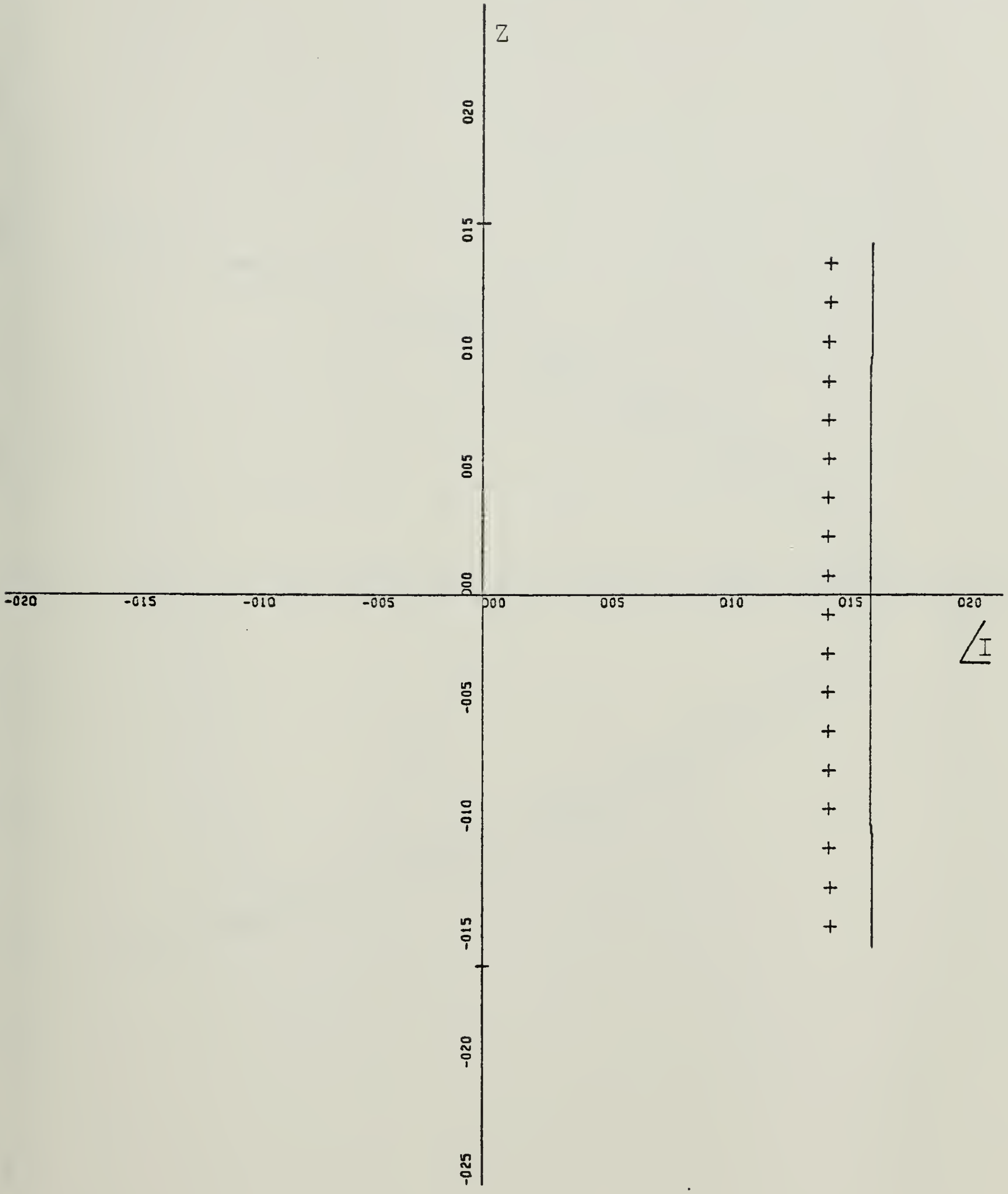
X-SCALE=5.00E-04 UNITS INCH.
Y-SCALE=5.00E-01 UNITS INCH.
CURRENT MAGNITUDE :FEM/T=SOLID :INT.EQ=X
ZL=.5:AL=0.01:BL=0.001:ALFA=90:NW=40



X-SCALE=5.00E+01 UNITS INCH.
Y-SCALE=5.00E-01 UNITS INCH.
CURRENT PHASE :FEM/T=SOLID :INT.EQ=PLUS
ZL=.5:AL=0.01:BL=0.001:ALFA=90:NW=40



X-SCALE=5.00E-04 UNITS INCH.
Y-SCALE=5.00E-01 UNITS INCH.
CURRENT MAGNITUDE :FEM/T=SOLID :INT.EQ=X
ZL=.5:AL=0.01:BL=0.0005:ALFA=90:NW=40



X-SCALE=5.00E+01 UNITS INCH.
Y-SCALE=5.00E-01 UNITS INCH.
CURRENT PHASE :FEM/T=SØLID :INT.EQ=PLUS
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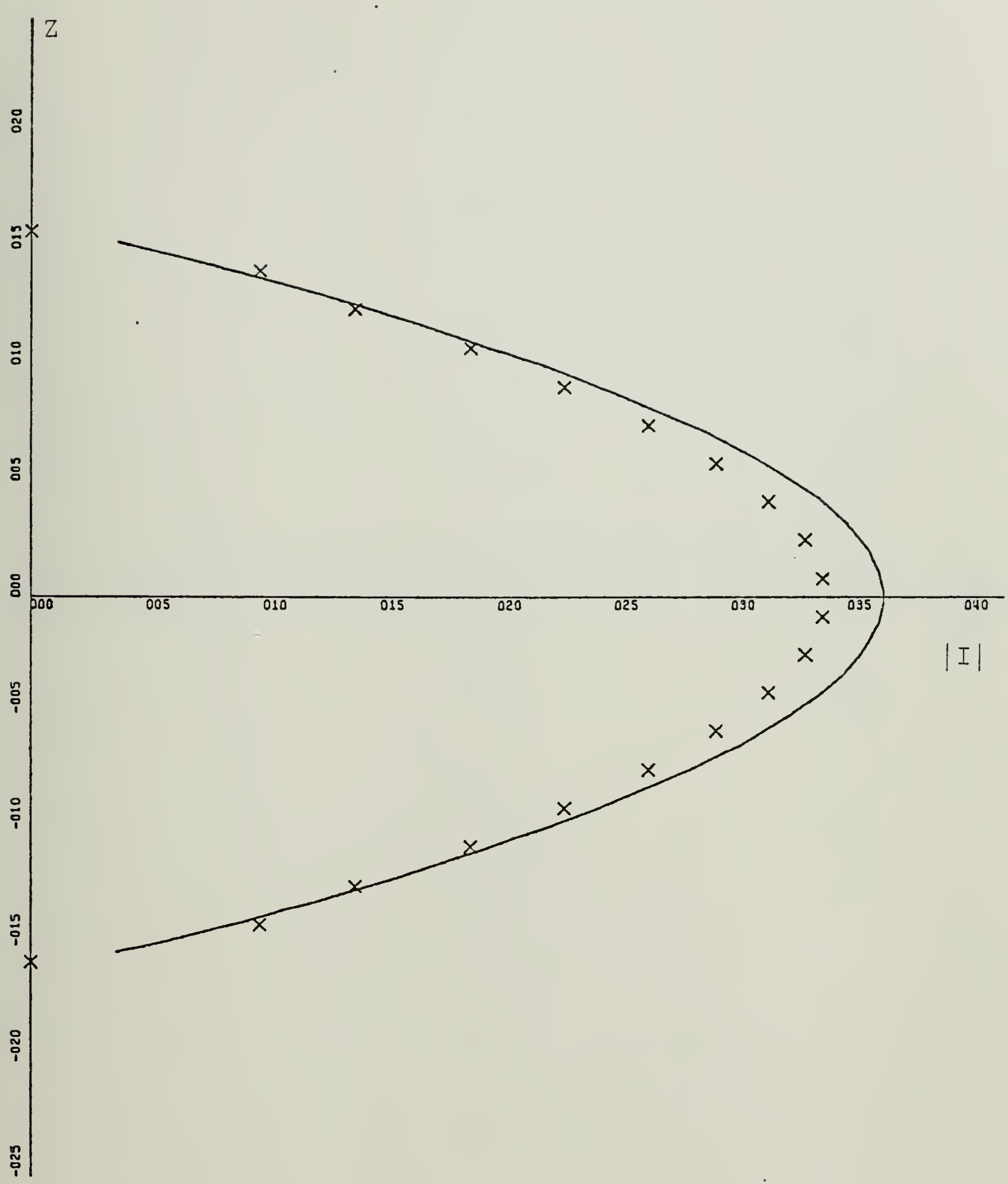
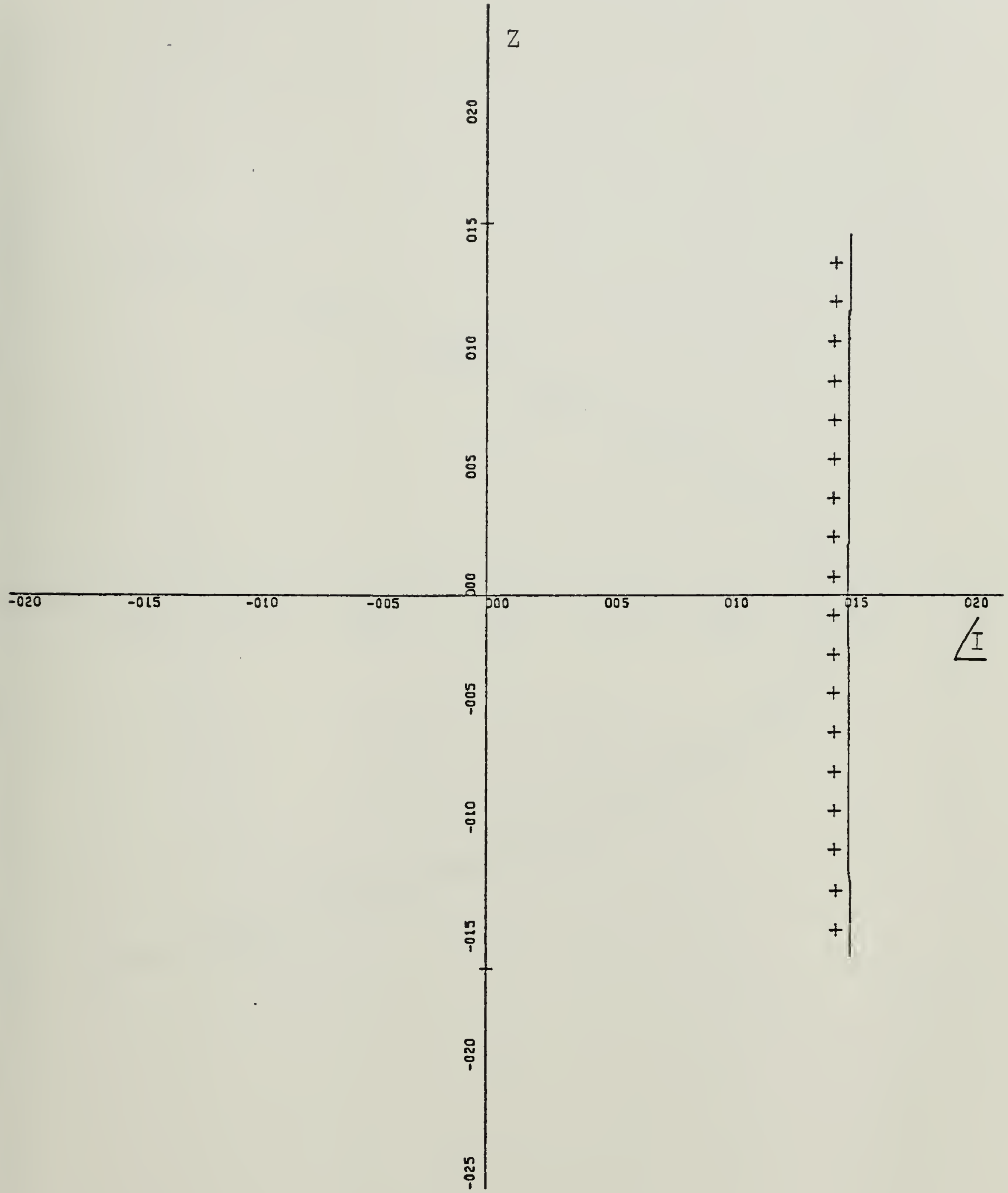
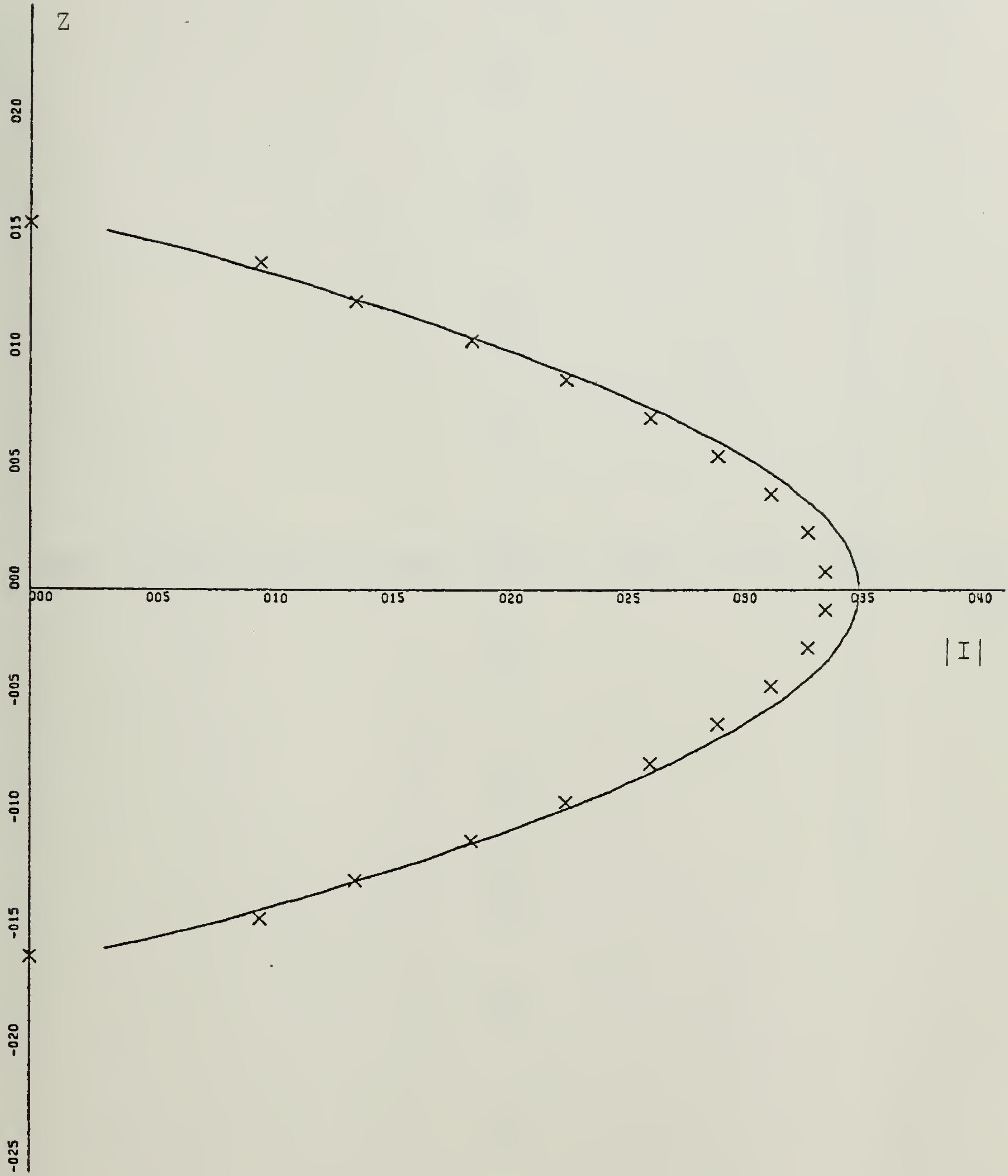


X-SCALE=5.00E-04 UNITS INCH.
Y-SCALE=5.00E-01 UNITS INCH.
CURRENT MAGNITUDE :FEM/T=SOLID :INT.EQ=X
ZL=.5:AL=0.01:BL=0.0005:ALFA=90:NW=70

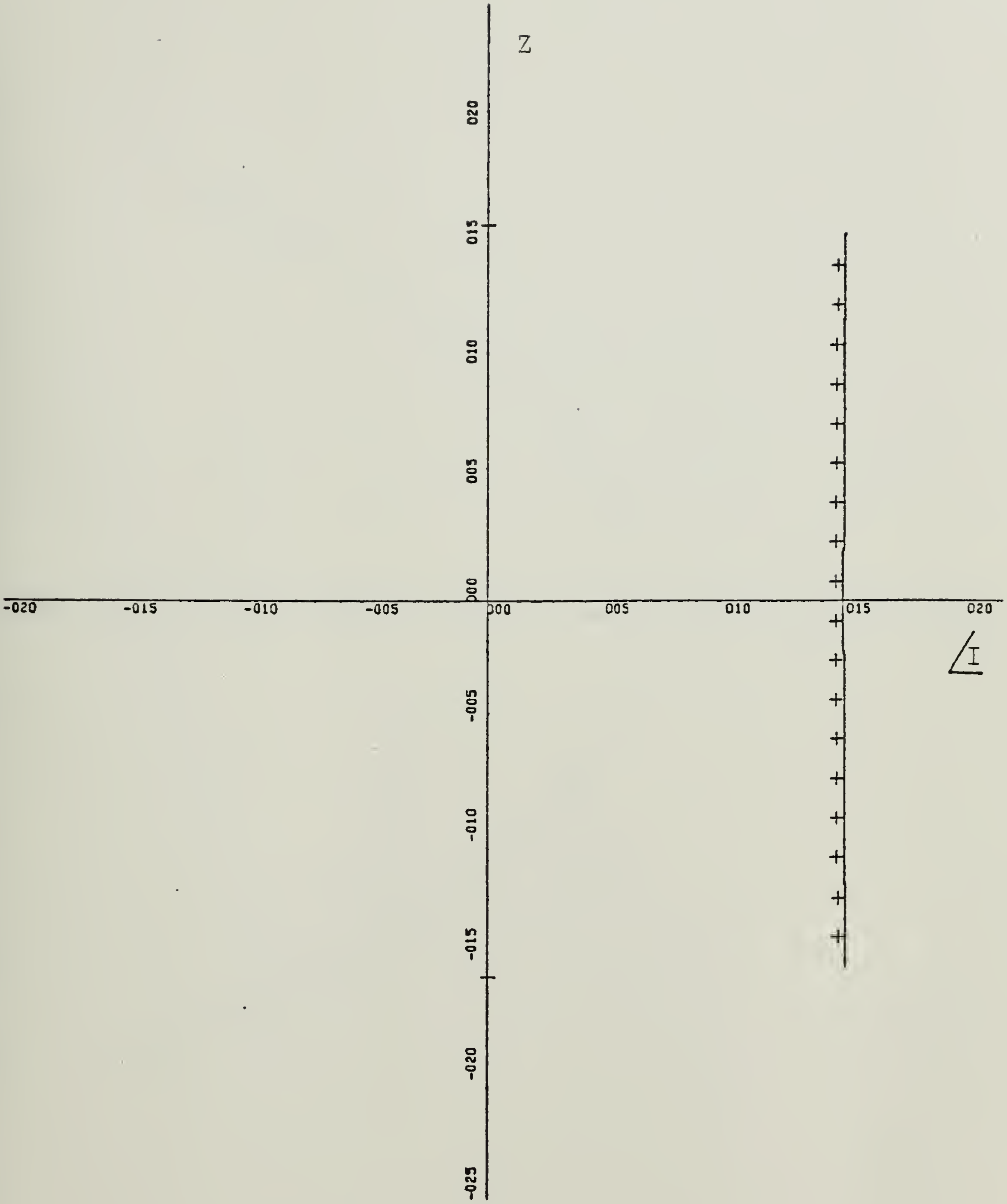


X-SCALE=5.00E+01 UNITS INCH.
Y-SCALE=5.00E-01 UNITS INCH.
CURRENT PHASE :FEM/T=SOLID :INT.EQ=PLUS
ZL=.5:AL=0.01:BL=0.0005:ALFA=90:NW=70

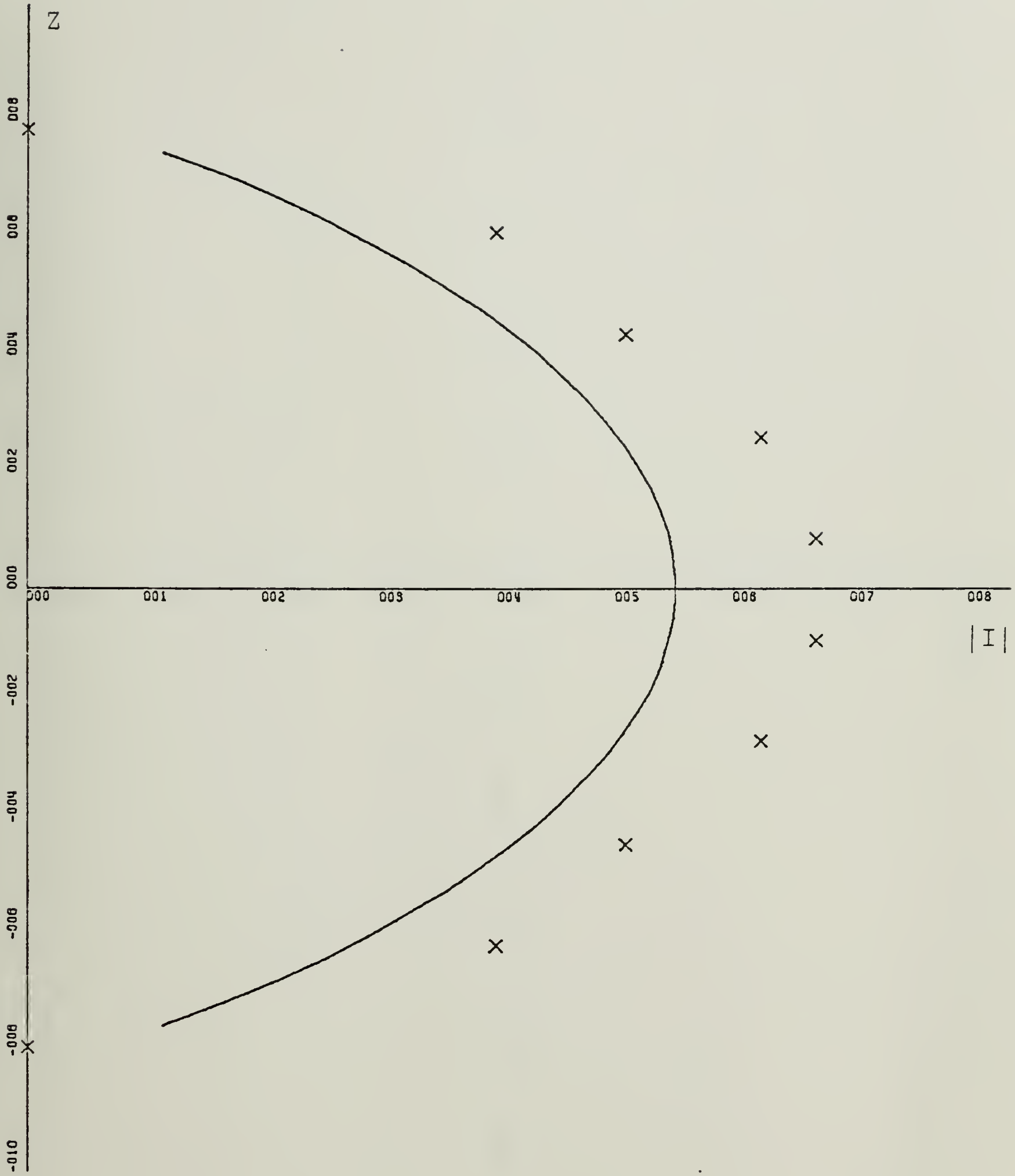


X-SCALE=5.00E-04 UNITS INCH.
Y-SCALE=5.00E-01 UNITS INCH.

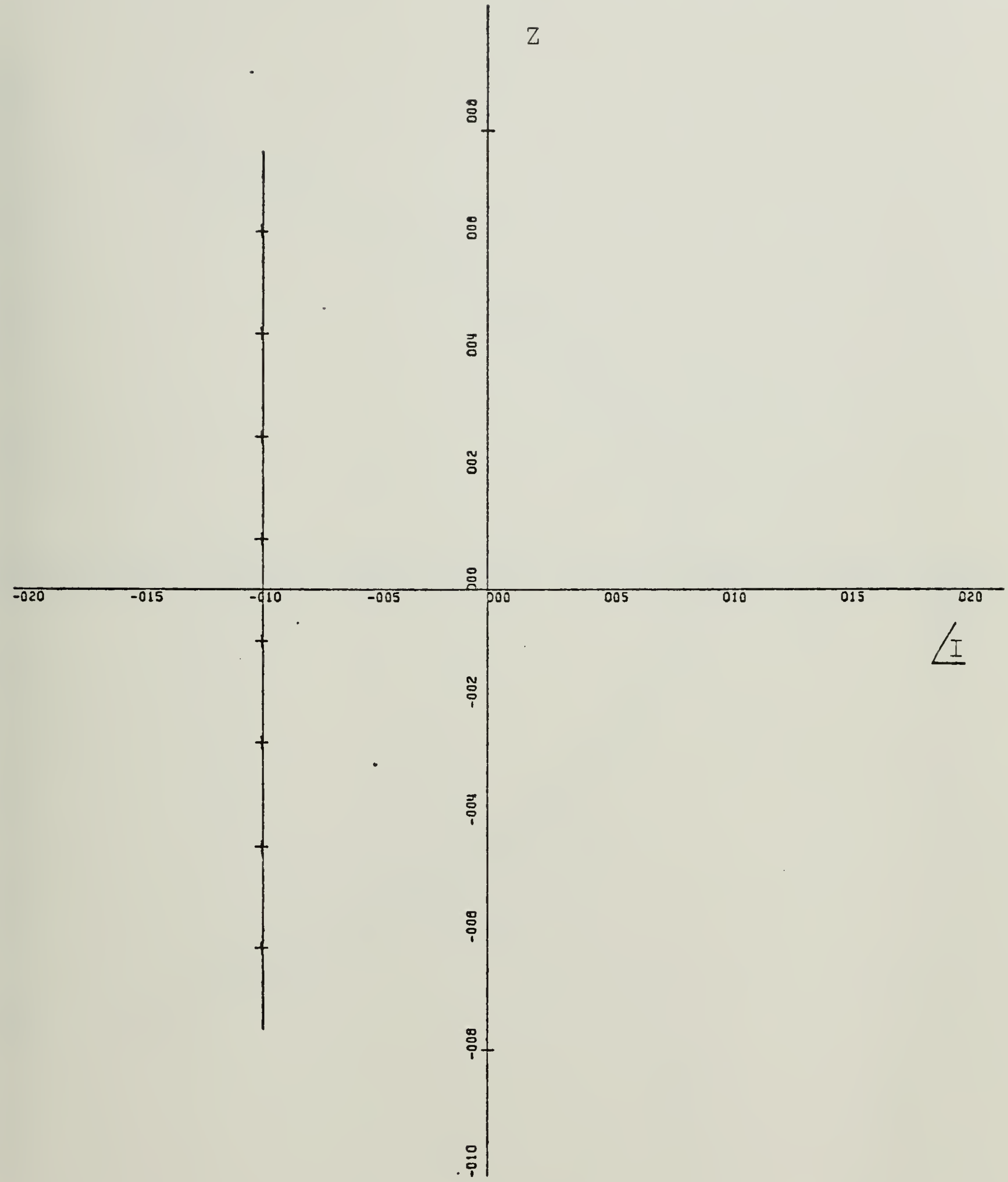
CURRENT MAGNITUDE :FEM/T=SOLID :INT.EQ=X
ZL=.5:AL=0.01:BL=0.0005:ALFA=90:NW=100



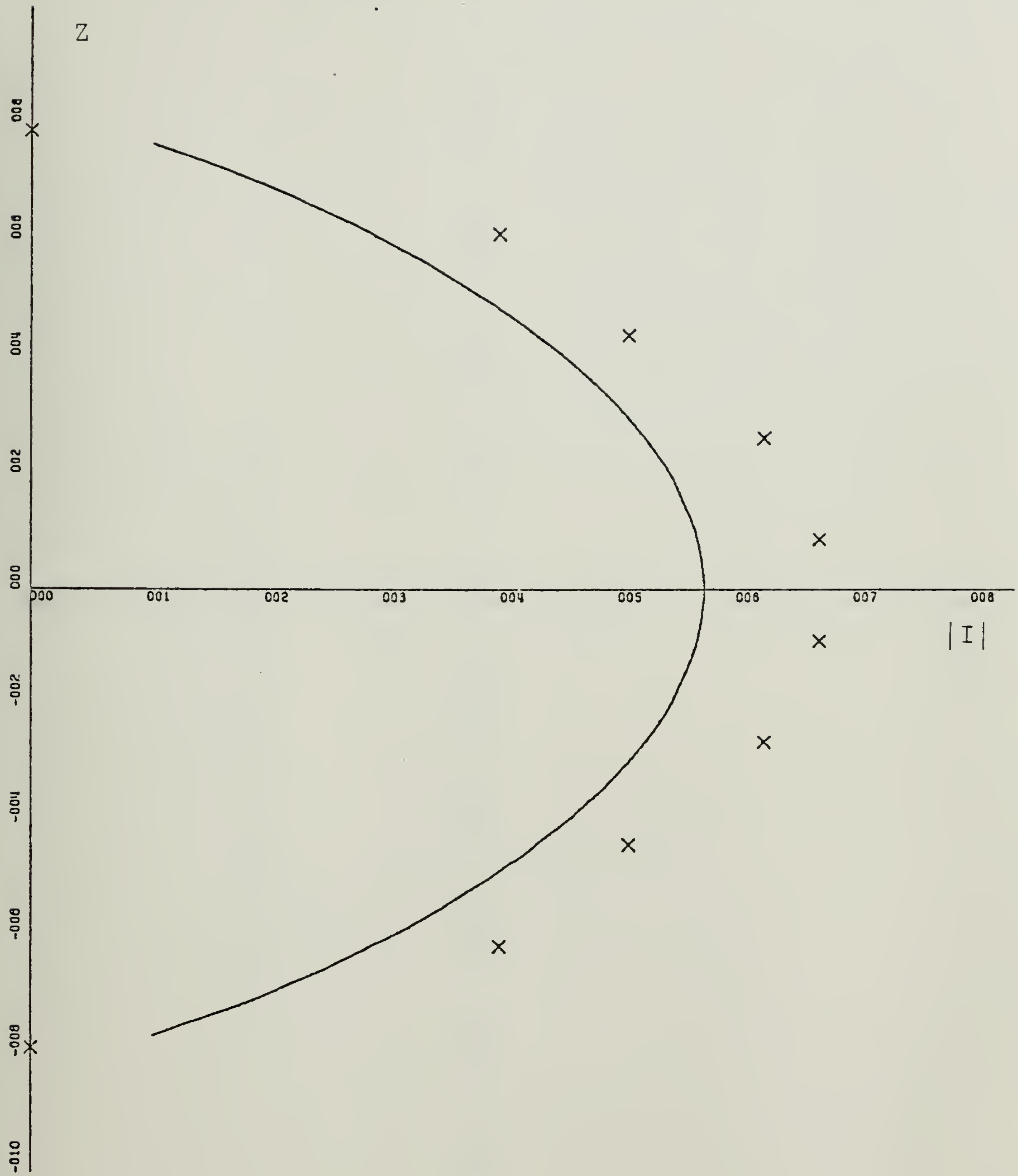
X-SCALE=5.00E+01 UNITS INCH.
Y-SCALE=5.00E-01 UNITS INCH.
CURRENT PHASE :FEM/T=SOLID :INT.EQ=PLUS
ZL=.5:AL=0.01:BL=0.0005:ALFA=90:NW=100



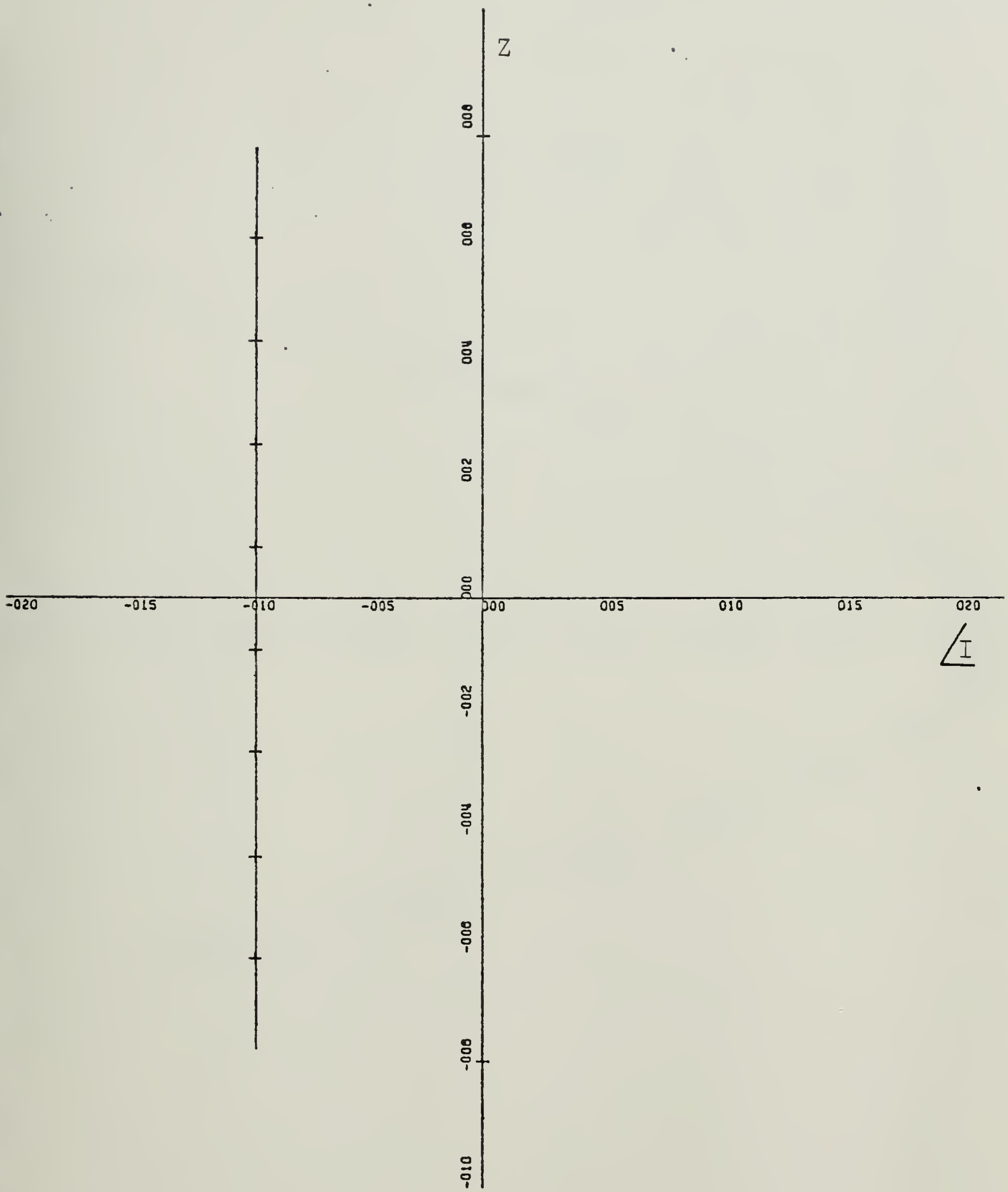
X-SCALE=1.00E-04 UNITS INCH.
Y-SCALE=2.00E-01 UNITS INCH.
CURRENT MAGNITUDE :FEM/T=SOLID :INT.EQ=X
ZL=.25:AL=0.01:BL=0.001:ALFA=90:NW=40



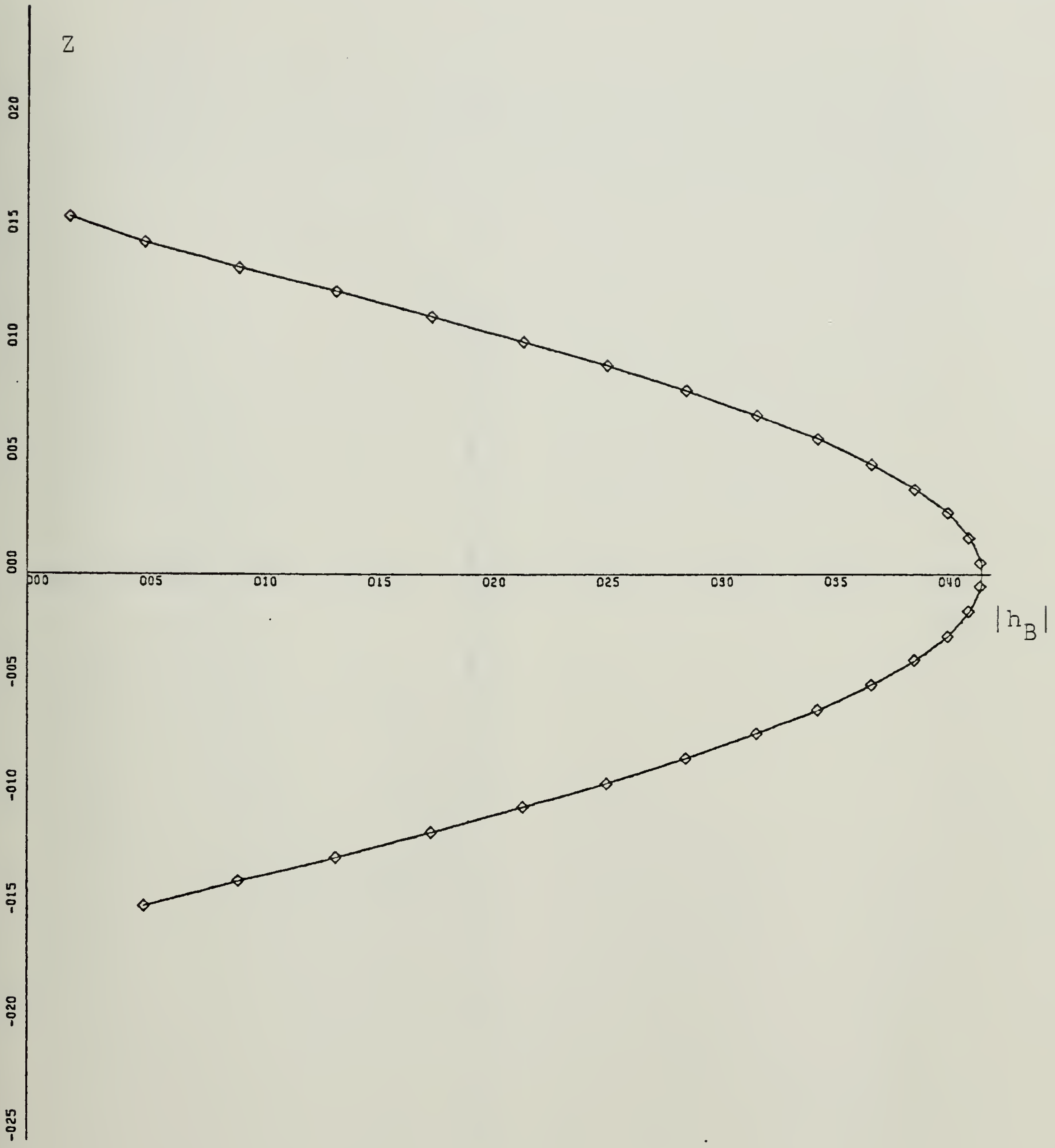
X-SCALE=5.00E+01 UNITS INCH.
Y-SCALE=2.00E-01 UNITS INCH.
CURRENT PHASE :FEM/T=SOLID :INT.EQ=PLUS
ZL=.25:AL=0.01:BL=0.001:ALFA=90:NW=40



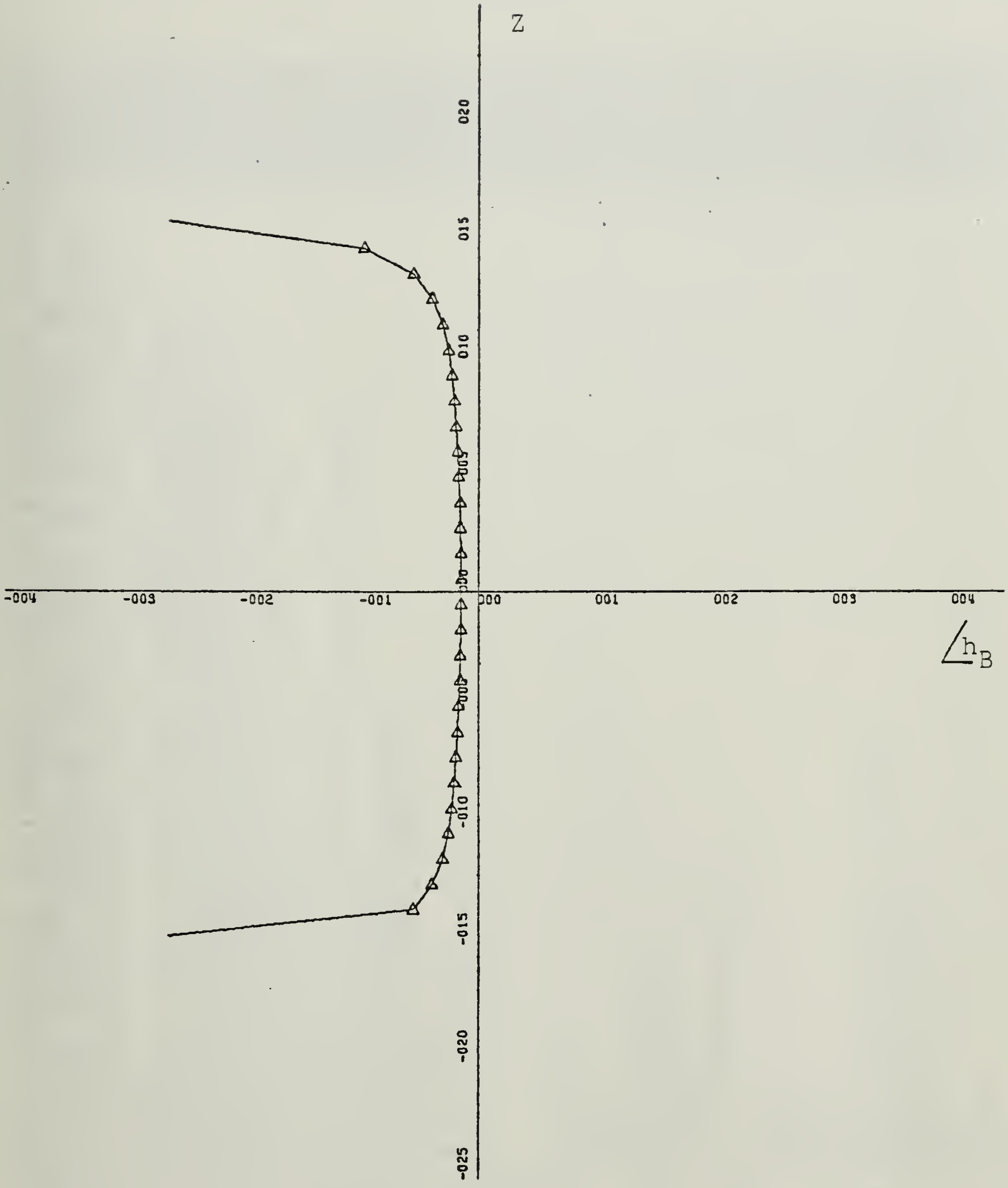
X-SCALE=1.00E-04 UNITS INCH.
Y-SCALE=2.00E-01 UNITS INCH.
CURRENT MAGNITUDE :FEM/T=SOLID :INT.EQ=X
ZL=.25:AL=0.01:BL=0.0005:ALFA=90:NW=70



X-SCALE=5.00E+01 UNITS INCH.
Y-SCALE=2.00E-01 UNITS INCH.
CURRENT PHASE :FEM/T=SOLID :INT.EQ=PLUS
ZL=.25:AL=0.01:BL=0.0005:ALFA=90:NW=70



X-SCALE=5.00E+02 UNITS INCH.
Y-SCALE=5.00E-01 UNITS INCH.
BOUNDARY FIELD MAG. : ZL=.5 : DR=DZ/4
SOLID=EXACT : DIAMONDS=T : AL=1/100



X-SCALE=1.00E+00 UNITS INCH.
Y-SCALE=5.00E-01 UNITS INCH.
BOUNDARY FIELD PHASE : ZL=.5 : DR=DZ/4
SOLID=EXACT : TRIANGLES=T : AL=1/100

APPENDIX D

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C     THIS PROGRAM CALCULATES THE INDUCED CURRENT ON A THIN WIRE IN FREEFFM00010
C     SPACE DUE TO AN INCIDENT PLANE WAVE. IT USES THE FEM AND THE      FFM00020
C     VECTOR POTENTIAL A. IT ALSO FINDS THE CURRENT BY HALLENS INTEGRAL FFM00030
C     EQUATION, AND BOTH ARE PLOTTED BY DRAWP (VERSATEC).A ROUTINE TO   FFM00040
C     CHECK THE T MATRIX (VECTOR POT.) IS INCLUDED BUT MAY BE DELETED   FFM00050
C     WITH NO HARM.                                                     FFM00060
C     TO USE , SPECIFY : NO. OF NODES=NW ; DIMENSIONS OF ARRAYS :       FFM00070
C     T(NB,NB),HW(NW),HI(NB),C(NB),A(NW,NW),B(NW,NB),HB(NB),HS(NB),     FFM00080
C     RP(NW),THETA(NW),ZP(NW),IEX(NW),RP1T(NB),ANGLE1(NB),RPT(NB),      FFM00090
C     ANGLE(NB),ZPB(NB),CMAG(NI),CFAZ(NI),CMAG(NI)  (NB=NW+1)           FFM00100
      COMPLEX F30,F21,F12,F20,F11,F10                                   FFM00110
      COMMON/TEN/F30,F21,F12,F20,F11,F10,P30,P21,P12,P20,P11,P10        FFM00120
      COMPLEX T(41,41),HW(40),HI(41),C(41),HB(41),HS(41)                FFM00130
      REAL RP(40),THETA(40),ZP(40),                                     FFM00140
     1A(40,40),B(40,41),                                                FFM00150
     1IEX(40),RP1T(41),ANGLE1(41),RPT(41),ANGLE(41),ZPB(41),            FFM00160
     1CMAG(20),CFAZ(20),ZC(20)                                          FFM00170
      COMPLEX UINT(3,3),LINT(3,3),DETERM                                FFM00180
      REAL RTB(28)/28*0.0/,R(3),Z(3)                                    FFM00190
      REAL*8 TITLE(12)                                                  FFM00200
      INTEGER ITB(12)/12*0/                                             FFM00210
      EQUIVALENCE (TITLE,RTB(5))                                        FFM00220
      MOR = 1                                                           FFM00230
      NW = 40                                                           FFM00240
      NI = 20                                                           FFM00250
      NB = NW+1                                                         FFM00260
      ALAM = 1.0/50.0                                                   FFM00270
      BLAM = 0.001                                                      FFM00280
      ZLAM = 0.5                                                        FFM00290
      DR = 6.283185*BLAM                                                FFM00300
      DZ = ZLAM*6.283185/NB                                             FFM00310
      RA = ALAM*6.283185                                                FFM00320
      DO 10 J = 1,NW                                                    FFM00330
      ZP(J) = DZ*(2.0*J-NB)/2.0                                         FFM00340
   10 CONTINUE                                                          FFM00350
      R(1) = RA                                                         FFM00360
      R(2) = RA+DR                                                      FFM00370
      R(3) = R(1)                                                       FFM00380
      Z(1) = DZ                                                         FFM00390
      Z(2) = 2.*DZ                                                      FFM00400
      Z(3) = Z(2)                                                       FFM00410
C     VARELA FINDS THE ELEMENT INTEGRALS                                FFM00420
      CALL VARELA (MOR,R,Z,UINT)                                        FFM00430
      R(1) = RA+DR                                                      FFM00440
      R(2) = RA                                                         FFM00450
      R(3) = R(1)                                                       FFM00460
      Z(1) = 2.*DZ                                                      FFM00470
      Z(2) = DZ                                                         FFM00480
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      Z(3) = Z(2)                                                       FFM00490
      CALL VARELA (MOR,R,Z,LINT)                                        FFM00500
C     ABLOAD FILLS THE A AND B MATRICES WITH THE ELEMENT INTEGRALS      FFM00510
      CALL ABLOAD (JINT,LINT,NW,NB,A,B)                                 FFM00520
C     GFINT FINDS THE T MATRIX BY GAUSSIAN INTEGRATION OF THE VECTOR    FFM00530
C     POTENTIAL A                                                       FFM00540
      CALL GFINT (NW,NB,DR,DZ,RA,T)                                     FFM00550
      RM = RA+DR                                                        FFM00560
      ZL = NB+DZ/2.0                                                    FFM00570
C     THIS ROUTINE FINDS A RADIATED MAGNETIC FIELD (DUE TO A GIVEN      FFM00580
C     SINUSOIDAL CURRENT) THIS FIELD IS TO BE COMPARED TO THE FIELD     FFM00590
C     PRODUCED BY HS=T*IEX                                              FFM00600
      CALL HIGEN (RM,DZ,NW,IEX,H3,NB,ZL)                                FFM00610
C     HS IS THE SCATTERED FIELD. (T MATRIX TEST)                        FFM00620
      DO 17 I=1,NB                                                      FFM00630
   17 HS(I) = (0.0,0.0)                                                 FFM00640
      DO 25 I = 1,N3                                                    FFM00650
      DO 30 J = 1,NW                                                    FFM00660
      HS(I) = HS(I)+T(I,J)*IEX(J)*60.0/ALAM                             FFM00670
   30 CONTINUE                                                          FFM00680
   25 CONTINUE                                                          FFM00690
C     CFORM YIELDS C=-(A+B*T)INV*B                                      FFM00700
      CALL CFORM (NW,NB,A,B,T,C,DETERM,COND)                            FFM00710
      DETMAG = CABS(DETERM)                                             FFM00720
      DO 77 II = 6,6                                                    FFM00730
      ALFA = (1.570798/6.0)*II                                          FFM00740
C     ZCURR CALCULATES THE INCIDENT PLANE WAVE HI AND GIVES HW=C*HI     FFM00750
      CALL ZCURR (HW,HI,RA,DR,DZ,ALFA,NW,NB,RP,THETA,T)                 FFM00760
C     INTEQ IS THE INTEGRAL EQUATION FORMULATION USED FOR COMPARISON    FFM00770
      CALL INTEQ (ZC,CMAG,CFAZ)                                         FFM00780
      DO 20 J = 1,NI                                                    FFM00790
   20 ZC(J) = 6.283185*ZC(J)                                            FFM00800
      ITB(1) = 1                                                        FFM00810
      ITB(2) = 1                                                        FFM00820
      ITB(3) = 9                                                        FFM00830
      ITB(8) = 2                                                        FFM00840
      ITB(9) = 5                                                        FFM00850
      ITB(10) = 0                                                       FFM00860
      RTB(1) = 0.0                                                      FFM00870
      READ (5,51) TITLE                                                 FFM00880
      CALL DRAWP (NI,CMAG,ZC,ITB,RTB)                                   FFM00890
      ITB(1) = 3                                                        FFM00900
      ITB(2) = 0                                                        FFM00910
      CALL DRAWP (NW,RP,ZP,ITB,RTB)                                     FFM00920
      ITB(1) = 1                                                        FFM00930
      ITB(2) = 2                                                        FFM00940
      ITB(10) = 2                                                       FFM00950
      ITB(11) = 4                                                       FFM00960
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      RTB(1) = 5.00E+01                                                 FFM00970
      READ (5,51) TITLE                                                 FFM00980
      CALL DRAWP (NI,CFAZ,ZC,ITB,RTB)                                   FFM00990
      ITB(1) = 3                                                        FFM01000
      ITB(2) = 0                                                        FFM01010
      CALL DRAWP (NW,THETA,ZP,ITB,RTB)                                  FFM01020
   51 FORMAT (6A8)                                                      FFM01030
      WRITE (6,300) (HI(I),I = 1,NB)                                    FFM01040
      WRITE (6,400) (RP(I),THETA(I),I=1,NW)                             FFM01050
      WRITE (6,100) ALAM,NW,DR,DZ,ZLAM,ALFA                             FFM01060
  100 FORMAT (' ALAM = ',E12.6,'    NW = ',I4,'    DR = ',E12.6/        FFM01070
     1'    DZ = ',E12.6,'    ZLAM = ',E12.6,'    ALFA = ',F7.2)         FFM01080
      WRITE(6,200) DETMAG,COND                                          FFM01090
  200 FORMAT(' DETMAG =  ',E12.6,' COND =  ',E12.6)                     FFM01100
  300 FORMAT ('1THE INCIDENT FIELD VECTOR IS '//100(' ',(E12.6,1X,      FFM01110
     1E12.6)/))                                                         FFM01120
  400 FORMAT ('OTHE POLAR FORM (RP,THETA) OF THE WIRE CURRENT IS '      FFM01130
     1//100(' ',(E12.6,2X,E12.6)/))                                     FFM01140
   77 CONTINUE                                                          FFM01150
      DO 40 I  = 1,N3                                                   FFM01160
      RP1T(I) = CABS(HB(I))                                             FFM01170
      ANGLE1(I) = ATAN2(AIMAG(HB(I)),REAL(HB(I)))*180.0/3.1415927       FFM01180
   40 CONTINUE                                                          FFM01190
      WRITE (6,500) (RP1T(I),ANGLE1(I),I=1,N3)                          FFM01200
  500 FORMAT ('OTHE EXACT BOUNDARY FIELD IS '//100(' ',(E12.6,          FFM01210
     12X,E12.6)/))                                                      FFM01220
      DO 50 I  = 1,N3                                                   FFM01230
      RPT(I) = CABS(HS(I))                                              FFM01240
      ANGLE(I) = ATAN2(AIMAG(HS(I)),REAL(HS(I)))*180.0/3.1415927        FFM01250
   50 CONTINUE                                                          FFM01260
      WRITE (6,600) (RPT(I),ANGLE(I),I = 1,NB)                          FFM01270
  600 FORMAT ('OTHE T MATRIX FIELD IS (RP,ANGLE) '//100(' ',(E12.6,2X,  FFM01280
     1E12.6)/))                                                         FFM01290
      DO 60 I  = 1,N3                                                   FFM01300
   60  ZPB(I) = DZ*(2.0*I-NB)/2.0                                       FFM01310
      ITB(1) = 1                                                        FFM01320
      ITB(2) = 0                                                        FFM01330
      ITB(8) = 2                                                        FFM01340
      ITB(9) = 5                                                        FFM01350
      ITB(10) = 0                                                       FFM01360
      READ (5,51) TITLE                                                 FFM01370
      CALL DRAWP (NB,RP1T,ZPB,ITB,RTB)                                  FFM01380
      ITB(1) = 3                                                        FFM01390
      ITB(2) = 4                                                        FFM01400
      CALL DRAWP(NB,RPT,ZPB,ITB,RTB)                                    FFM01410
      ITB(1) = 1                                                        FFM01420
      ITB(2) = 0                                                        FFM01430
      ITB(8) = 2                                                        FFM01440
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      ITB(9) = 5                                                        FFM01450
      ITB(10) = 2                                                       FFM01460
      READ (5,51) TITLE                                                 FFM01470
      CALL DRAWP (NB,ANGLE1,ZPB,ITB,RTB)                                FFM01480
      ITB(1) = 3                                                        FFM01490
      ITB(2) = 5                                                        FFM01500
      CALL DRAWP(NW,ANGLE,ZPB,ITB,RTB)                                  FFM01510
      STOP                                                              FFM01520
      END                                                               FFM01530
C                                                                       FFM01540
C                                                                       FFM01550
C                                                                       FFM01560
      SUBROUTINE ABLOAD (U,L,NW,NB,A,B)                                 FFM01570
      COMPLEX U(3,3),L(3,3)                                             FFM01580
      REAL A(NW,NW),B(NW,NB)                                            FFM01585
      BETA = U(1,1)+U(3,3)+L(2,2)                                       FFM01590
      P1 = U(3,2)+L(3,2)                                                FFM01600
      P2 = U(1,2)+L(1,2)                                                FFM01610
      DO 5 K = 1,NW                                                     FFM01620
      DO 7 J = 1,NW                                                     FFM01630
      A(K,J) = (0.0)                                                    FFM01640
      B(K,J) = (0.0)                                                    FFM01650
    7 CONTINUE                                                          FFM01660
    5 CONTINUE                                                          FFM01670
      DO 8 J = 1,NW                                                     FFM01680
      B(J,NW+1) = (0.0)                                                 FFM01690
    8 CONTINUE                                                          FFM01700
      DO 10 K = 1,NW                                                    FFM01710
      A(K,K) = BETA                                                     FFM01720
   10 CONTINUE                                                          FFM01730
      DO 20 K = 2,NW                                                    FFM01740
      A(K-1,K) = U(1,3)                                                 FFM01750
      A(K,K-1) = U(1,3)                                                 FFM01760
   20 CONTINUE                                                          FFM01770
      DO 30 K = 1,NW                                                    FFM01780
      B(K,K) = P1                                                       FFM01790
      B(K,K+1) = P2                                                     FFM01800
   30 CONTINUE                                                          FFM01810
      RETURN                                                            FFM01820
      END                                                               FFM01830
C                                                                       FFM01840
C                                                                       FFM01850
C                                                                       FFM01860
      SUBROUTINE GFINT(NW,NB,DR,DZ,RA,T)                                FFM01870
      COMPLEX T(NB,NB)                                                  FFM01880
      COMPLEX F1,F2,LF11,LF12,LF21,LF22,F10,F20                         FFM01890
      REAL W(4),X(4)                                                    FFM01900
      DO 5 K =1,NB                                                      FFM01910
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      DO 7 L =1,NW                                                      FFM01920
      T(K,L) = (0.0,0.0)                                                FFM01930
    7 CONTINUE                                                          FFM01940
    5 CONTINUE                                                          FFM01950
      RM = RA+DR                                                        FFM01960
      W(1) = 0.36268378                                                 FFM01970
      W(2) = 0.31370665                                                 FFM01980
      W(3) = 0.22238103                                                 FFM01990
      W(4) = 0.10122854                                                 FFM02000
      X(1) = 0.18343464                                                 FFM02010
      X(2) = 0.52553241                                                 FFM02020
      X(3) = 0.79666648                                                 FFM02030
      X(4) = 0.96023986                                                 FFM02040
      DO 10 M= 1,NB                                                     FFM02050
      DO 20 N= 1,NW                                                     FFM02060
      F10 = (0.0,0.0)                                                   FFM02070
      F20 = (0.0,0.0)                                                   FFM02080
      DO 30 NG= 1,4                                                     FFM02090
      LF11 = F1  (X(NG),N,M,NW,DR,DZ,RA)                                FFM02100
      LF12 = F1  (-X(NG),N,M,NW,DR,DZ,RA)                               FFM02110
      LF21 = F2  (X(NG),N,M,NW,DR,DZ,RA)                                FFM02120
      LF22 = F2  (-X(NG),N,M,NW,DR,DZ,RA)                               FFM02130
      F10 = F10+W(NG)*(LF11+LF12)                                       FFM02140
      F20 = F20+W(NG)*(LF21+LF22)                                       FFM02150
   30 CONTINUE                                                          FFM02160
      T(M,N) = 0.5*RA*RM*( F10+F20)                                     FFM02170
   20 CONTINUE                                                          FFM02180
   10 CONTINUE                                                          FFM02190
      RETURN                                                            FFM02200
      END                                                               FFM02210
C                                                                       FFM02220
      FUNCTION F1(X,N,M,NW,DR,DZ,RA)                                    FFM02230
      COMPLEX F1,EJR,GMZ                                                FFM02240
      WZ = (X+1.)/2.                                                    FFM02250
      ZNM1 = DZ*(2.*N-NW-3.)/2.                                         FFM02260
      ZM = DZ*(2.*M-NW-1.)/2.                                           FFM02270
      ZX = ZNM1+DZ*WZ                                                   FFM02280
      RM = RA + DR                                                      FFM02290
      RMZ = SQRT((ZX-ZM)**2+RM*RM)                                      FFM02300
      EJR = CMPLX(COS(RMZ),-SIN(RMZ))                                   FFM02310
      GMZ = EJR*CMPLX(1./RMZ,1.)/(RMZ*RMZ)                              FFM02320
      F1 = DZ*WZ*GMZ/2.                                                 FFM02330
      RETURN                                                            FFM02340
      END                                                               FFM02350
C                                                                       FFM02360
      FUNCTION F2(X,N,M,NW,DR,DZ,RA)                                    FFM02370
      COMPLEX F2,EJR,GMZ                                                FFM02380
      WZ = (1.-X)/2.                                                    FFM02390
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      ZN = DZ*(2.*N-NW-1.)/2.                                           FFM02400
      ZM = DZ*(2.*M-NW-1.)/2.                                           FFM02410
      RM = RA+DR                                                        FFM02420
      ZX = ZN+DZ*(X+1.)/2.                                              FFM02430
      RMZ = SQRT((ZX-ZM)**2+RM*RM)                                      FFM02440
      EJR = CMPLX(COS(RMZ),-SIN(RMZ))                                   FFM02450
      GMZ = EJR*CMPLX(1./RMZ,1.)/(RMZ*RMZ)                              FFM02460
      F2 = DZ*WZ*GMZ/2.                                                 FFM02470
      RETURN                                                            FFM02480
      END                                                               FFM02490
C                                                                       FFM02500
C                                                                       FFM02510
C                                                                       FFM02520
      SUBROUTINE CFORM (NW,NB,A,B,T,C,DETERM,COND)                      FFM02530
C     TO SAVE CORE,T IS USED AS A WORKING SPACE. MATRIX MULTIPLICATION  FFM02540
C     IS DONE USING ONE ROW (COLUMN) OF T AT A TIME. RESULTS ARE  STOREDFFM02550
C     IN C,THEN PUT IN THE NO LONGER NEEDED ROW (COLUMN) OF T           FFM02560
      COMPLEX T(NB,N3),C(NB),DETERM                                     FFM02570
      REAL A(NW,NW),B(NW,NB)                                            FFM02575
      DO 50 J = 1,NW                                                    FFM02580
      DO 1 K = 1,NW                                                     FFM02590
    1 C(K) = (0.0,0.0)                                                  FFM02600
      DO 30 I = 1,NW                                                    FFM02610
      DO 30 L = 1,N3                                                    FFM02620
   30 C(I) = C(I)+B(I,L)*T(L,J)                                         FFM02630
      DO 40 K = 1,NW                                                    FFM02640
   40 T(K,J) = C(K)                                                     FFM02650
  50  CONTINUE                                                          FFM02660
      DO 60 I = 1,NW                                                    FFM02670
      DO 70 J = 1,NW                                                    FFM02680
      T(I,J) = A(I,J)+T(I,J)                                            FFM02690
  70  CONTINUE                                                          FFM02700
  60  CONTINUE                                                          FFM02710
      CALL CSMINV  (T,NB,NW,DETERM,COND,0)                              FFM02720
C      T IS NOW THE INVERSE OF  T IN LINE 14                            FFM02730
      DO 90 J = 1,NW                                                    FFM02740
      DO 2 K = 1,NB                                                     FFM02750
    2 C(K) = (0.0,0.0)                                                  FFM02760
      DO 80 I = 1,N3                                                    FFM02770
      DO  80 L = 1,NW                                                   FFM02780
   80 C(I) = C(I)-T(J,L)*B(L,I)                                         FFM02790
      DO 85 K = 1,N3                                                    FFM02800
   85 T(J,K) = C(K)                                                     FFM02810
  90  CONTINUE                                                          FFM02820
      RETURN                                                            FFM02830
      END                                                               FFM02840
C                                                                       FFM02850
C                                                                       FFM02860
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C                                                                       FFM02870
      SUBROUTINE ZCURR (HW,HI,RA,DR,DZ,ALFA,NW,NB,RP,THETA,T)          FFM02880
      COMPLEX T(NB,NB),HW(NW),HI(NB),ALFDEP                             FFM02890
      REAL RP(NW),THETA(NW)                                             FFM02900
      DO 10 I = 1,NW                                                    FFM02910
      HW(I) = (0.0,0.0)                                                 FFM02920
   10 CONTINUE                                                          FFM02930
      RM = RA+DR                                                        FFM02940
      X = RM*SIN(ALFA)                                                  FFM02950
      ADJO = -0.5*X+(1./16.)*X**3-(1./384.)*X**5                        FFM02960
      DO 20 I = 1,NB                                                    FFM02970
      ZNB = DZ*(2.*I-NB)/2.0                                            FFM02980
      ZC = ZNB*COS(ALFA)                                                FFM02990
      ALFDEP = CMPLX(SIN(ZC),COS(ZC))                                   FFM03000
      HI(I) = ALFDEP*ADJO                                               FFM03010
   20 CONTINUE                                                          FFM03020
      DO 30 I = 1,NW                                                    FFM03030
      DO 40 J = 1,NB                                                    FFM03040
      HW(I) = HW(I)+T(I,J)*HI(J)                                        FFM03050
   40 CONTINUE                                                          FFM03060
   30 CONTINUE                                                          FFM03070
      DO 50 I = 1,NW                                                    FFM03080
      RP(I) = CABS(HW(I))*RA/(120.0*3.1415927)                          FFM03090
      THETA(I) = ATAN2(AIMAG(HW(I)),REAL(HW(I)))*180.0/3.1415927        FFM03100
   50 CONTINUE                                                          FFM03110
      RETURN                                                            FFM03120
      END                                                               FFM03130
C                                                                       FFM03140
C                                                                       FFM03150
C                                                                       FFM03160
      SUBROUTINE CSMINV  (A,NDIM,N,DETERM,COND,INORM)                   FFM03170
C           INORM - FLAG TO NORMALIZE COLUMNS AND ROWS OF MATRIX A      FFM03180
C       MATRIX NORMALIZATION BY M.A. MORGAN             APRIL 24,1978  FFM03190
      COMPLEX A(NDIM,NDIM),PIVOT(100),AMAX,T,SWAP,DETERM,U              FFM03200
      INTEGER IPIVOT(100),INDEX(100,2)                                  FFM03210
      REAL    TEMP,ALPHA(100),COL(100),ROW(100)                         FFM03220
      IF(INORM.NE.1) GO TO 7                                            FFM03230
        DO 3 K=1,N                                                      FFM03240
        COL(K)=0.0                                                      FFM03250
        DO 1 J=1,N                                                      FFM03260
        AJK=CABS(A(J,K))                                                FFM03270
        IF(AJK.GT.COL(K)) COL(K)=AJK                                    FFM03280
    1   CONTINUE                                                        FFM03290
        DO 2 J=1,N                                                      FFM03300
    2   A(J,K)=A(J,K)/COL(K)                                            FFM03310
    3   CONTINUE                                                        FFM03320
C         ROW NORMALIZING                                               FFM03330
        DO 6 J=1,N                                                      FFM03340
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        ROW(J)=0.0                                                      FFM03350
        DO 4 K=1,N                                                      FFM03360
        AJK=CABS(A(J,K))                                                FFM03370
        IF(AJK.GT.ROW(J)) ROW(J)=AJK                                    FFM03380
    4   CONTINUE                                                        FFM03390
        DO 5 K=1,N                                                      FFM03400
    5   A(J,K)=A(J,K)/ROW(J)                                            FFM03410
    6   CONTINUE                                                        FFM03420
    7 CONTINUE                                                          FFM03430
      DETERM =  CMPLX(1.0,0.0)                                          FFM03440
      SUMAXA=0.                                                         FFM03450
      DO 20 J=1,N                                                       FFM03460
      ALPHA(J)=0.0                                                      FFM03470
      SUMROW=0.                                                         FFM03480
      DO 10 I=1,N                                                       FFM03490
      ALPHA(J)=ALPHA(J)+A(J,I)* CONJG(A(J,I))                           FFM03500
   10 SUMROW=SUMROW + CABS(A(J,I))                                      FFM03510
      ALPHA(J)= SQRT(ALPHA(J))                                          FFM03520
      IF(SUMROW.GT.SUMAXA) SUMAXA=SUMROW                                FFM03530
   20 IPIVOT(J)=0                                                       FFM03540
      DO 600 I=1,N                                                      FFM03550
      AMAX=CMPLX(0.0,0.0)                                               FFM03560
      DO 105 J=1,N                                                      FFM03570
      IF (IPIVOT(J)-1) 60, 105, 60                                      FFM03580
   60 DO 100 K=1,N                                                      FFM03590
      IF (IPIVOT(K)-1) 80, 100, 740                                     FFM03600
  80  TEMP=AMAX* CONJG(AMAX)-A(J,K)* CONJG(A(J,K))                      FFM03610
      IF(TEMP)85,85,100                                                 FFM03620
   85 IROW=J                                                            FFM03630
      ICOLUM=K                                                          FFM03640
      AMAX=A(J,K)                                                       FFM03650
  100 CONTINUE                                                          FFM03660
  105 CONTINUE                                                          FFM03670
      IPIVOT(ICOLUM)=IPIVOT(ICOLUM)+1                                   FFM03680
      IF (IROW-ICOLUM) 140, 260, 140                                    FFM03690
  140 DETERM=-DETERM                                                    FFM03700
      DO 200 L=1,N                                                      FFM03710
      SWAP=A(IROW,L)                                                    FFM03720
      A(IROW,L)=A(ICOLUM,L)                                             FFM03730
  200 A(ICOLUM,L)=SWAP                                                  FFM03740
      SWAP=ALPHA(IROW)                                                  FFM03750
      ALPHA(IROW)=ALPHA(ICOLUM)                                         FFM03760
      ALPHA(ICOLUM)=SWAP                                                FFM03770
  260 INDEX(I,1)=IROW                                                   FFM03780
      INDEX(I,2)=ICOLUM                                                 FFM03790
      PIVOT(I)=A(ICOLUM,ICOLUM)                                         FFM03800
      U = PIVOT(I)                                                      FFM03810
      DETERM = DETERM*U                                                 FFM03820
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      DETERM=DETERM/ALPHA(ICOLUM)                                       FFM03830
      TEMP=PIVOT(I)* CONJG(PIVOT(I))                                    FFM03840
      IF(TEMP)330,720,330                                               FFM03850
  330 A(ICOLUM,ICOLUM) =  CMPLX(1.0,0.0)                                FFM03860
      DO 350 L=1,N                                                      FFM03870
      U = PIVOT(I)                                                      FFM03880
  350 A(ICOLUM,L) = A(ICOLUM,L)/J                                       FFM03890
 380  DO 550 L1=1,N                                                     FFM03900
      IF(L1-ICOLUM) 400, 550, 400                                       FFM03910
  400 T=A(L1,ICOLUM)                                                    FFM03920
      A(L1,ICOLUM)= CMPLX(0.0,0.0)                                      FFM03930
      DO 450 L=1,N                                                      FFM03940
      U = A(ICOLUM,L)                                                   FFM03950
  450 A(L1,L) = A(L1,L)-U*T                                             FFM03960
  550 CONTINUE                                                          FFM03970
600   CONTINUE                                                          FFM03980
 620  DO 710 I=1,N                                                      FFM03990
      L=N+1-I                                                           FFM04000
      IF (INDEX(L,1)-INDEX(L,2)) 630, 710, 630                          FFM04010
  630 JROW=INDEX(L,L)                                                   FFM04020
      JCOLUM=INDEX(L,2)                                                 FFM04030
      DO 705 K=1,N                                                      FFM04040
      SWAP=A(K,JROW)                                                    FFM04050
      A(K,JROW)=A(K,JCOLUM)                                             FFM04060
      A(K,JCOLUM)=SWAP                                                  FFM04070
  705 CONTINUE                                                          FFM04080
  710 CONTINUE                                                          FFM04090
      SUMAXI=0.                                                         FFM04100
      DO 910 I=1,N                                                      FFM04110
      SUMROW=0.                                                         FFM04120
      DO 900 J=1,N                                                      FFM04130
  900 SUMROW=SUMROW + CABS(A(I,J))                                      FFM04140
      IF(SUMROW.GT.SUMAXI) SUMAXI=SUMROW                                FFM04150
  910 CONTINUE                                                          FFM04160
      COND =  SUMAXA*SUMAXI                                             FFM04170
      IF(INORM.NE.1) GO TO 955                                          FFM04180
      DO 950 K=1,N                                                      FFM04190
      DO 950 J=1,N                                                      FFM04200
  950 A(J,K)=A(J,K)/(ROW(K)*COL(J))                                     FFM04210
  955 CONTINUE                                                          FFM04220
      RETURN                                                            FFM04230
  720 WRITE(6,730)                                                      FFM04240
  730 FORMAT('0',10('*******')/'0MATRIX IS SINGULAR'/'0',10('*******')) FFM04250
  740 RETURN                                                            FFM04260
        END                                                             FFM04270
C                                                                       FFM04280
C                                                                       FFM04290
C                                                                       FFM04300
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      SUBROUTINE VARELA (MOR,X,Y,H2)                                    FFM04310
      COMPLEX F30,F21,F12,F20,F11,F10                                   FFM04320
      COMMON/TEN/F30,F21,F12,F20,F11,F10,P30,P21,P12,P20,P11,P10        FFM04330
      COMPLEX H2(3,3),A(3,3),C2                                         FFM04340
      REAL T(3,3),X(3),Y(3)                                             FFM04350
      C2=(1.0,0.0)                                                      FFM04360
      RM=0.0                                                            FFM04370
      Q=((X(2)*Y(3)+X(3)*Y(1)+X(1)*Y(2))                                FFM04380
     1  -(X(3)*Y(2)+X(1)*Y(3)+X(2)*Y(1)))/2.0                           FFM04390
      T(1,1)=Y(2)-Y(3)                                                  FFM04400
      T(1,2)=Y(3)-Y(1)                                                  FFM04410
      T(1,3)=Y(1)-Y(2)                                                  FFM04420
      T(2,1)=X(3)-X(2)                                                  FFM04430
      T(2,2)=X(1)-X(3)                                                  FFM04440
      T(2,3)=X(2)-X(1)                                                  FFM04450
      T(3,1)=X(2)*Y(3)-X(3)*Y(2)                                        FFM04460
      T(3,2)=X(3)*Y(1)-X(1)*Y(3)                                        FFM04470
      T(3,3)=X(1)*Y(2)-X(2)*Y(1)                                        FFM04480
      DO 1 J=1,3                                                        FFM04490
      DO 1 K=1,3                                                        FFM04500
    1 T(J,K)=T(J,K)/(2.0*Q)                                             FFM04510
      CALL ARINT (RM,C2,X,Y,MOR)                                        FFM04520
      A(1,1)=4.*F30-P30                                                 FFM04530
      A(2,2)=F30+F12-P12                                                FFM04540
      A(3,3)=F10-P10                                                    FFM04550
      A(2,1)=2.*F21-P21                                                 FFM04560
      A(1,2)=A(2,1)                                                     FFM04570
      A(3,1)=2.*F20-P20                                                 FFM04580
      A(1,3)=A(3,1)                                                     FFM04590
      A(3,2)=F11-P11                                                    FFM04600
      A(2,3)=A(3,2)                                                     FFM04610
      DO 5 KND=1,3                                                      FFM04620
      DO 3 K=1,3                                                        FFM04630
      H2(KND,K)=(0.,0.)                                                 FFM04640
      DO 2 L=1,3                                                        FFM04650
      DO 2 J=1,3                                                        FFM04660
      H2(KND,K)=H2(KND,K)+T(J,KND)*A(J,L)*T(L,K)                        FFM04670
    2 CONTINUE                                                          FFM04680
    3 CONTINUE                                                          FFM04690
    5 CONTINUE                                                          FFM04700
      RETURN                                                            FFM04710
      END                                                               FFM04720
C                                                                       FFM04730
      SUBROUTINE ARINT (RM,C2,X,Y,MOR)                                  FFM04740
      COMPLEX F30,F21,F12,F20,F11,F10                                   FFM04750
      COMMON/TEN/F30,F21,F12,F20,F11,F10,P30,P21,P12,P20,P11,P10        FFM04760
      COMPLEX F(4),C,C2,CSR                                             FFM04770
      REAL X(3),Y(3),X1(3),X2(3),Y1(3),Y2(3),P(4)                       FFM04780
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      RM2=RM*RM                                                         FFM04790
      C=CSR(C2)                                                         FFM04800
      X1(3)=X(1)                                                        FFM04810
      X2(2)=X(1)                                                        FFM04820
      Y1(3)=Y(1)                                                        FFM04830
      Y2(2)=Y(1)                                                        FFM04840
      X1(1)=X(2)                                                        FFM04850
      X2(3)=X(2)                                                        FFM04860
      Y1(1)=Y(2)                                                        FFM04870
      Y2(3)=Y(2)                                                        FFM04880
      X1(2)=X(3)                                                        FFM04890
      X2(1)=X(3)                                                        FFM04900
      Y1(2)=Y(3)                                                        FFM04910
      Y2(1)=Y(3)                                                        FFM04920
      DI=1.0-2.*MOR                                                     FFM04930
      P30=0.0                                                           FFM04940
      P21=0.                                                            FFM04950
      P12=0.                                                            FFM04960
      P20=0.                                                            FFM04970
      P11=0.                                                            FFM04980
      P10=0.                                                            FFM04990
      F30=(0.,0.)                                                       FFM05000
      F21=(0.,0.)                                                       FFM05010
      F12=(0.,0.)                                                       FFM05020
      F20=(0.,0.)                                                       FFM05030
      F11=(0.,0.)                                                       FFM05040
      F10=(0.,0.)                                                       FFM05050
C       CHANGES 8/15/78 TO HANDLE DEC 10 ROUNDOFF                       FFM05060
      DO 33 L=1,3                                                       FFM05070
      XS=X2(L)+X1(L)                                                    FFM05080
      IF(XS.LT.1.0E-4) GO TO 22                                         FFM05090
      DX=X2(L)-X1(L)                                                    FFM05100
      DY=Y2(L)-Y1(L)                                                    FFM05110
      IF(ABS(DX/XS).LT.1.0E-5) GO TO 22                                 FFM05120
      A=DY/DX                                                           FFM05130
      B=Y1(L)-A*X1(L)                                                   FFM05140
      A2=A*A                                                            FFM05150
      A3=A*A2                                                           FFM05160
      B2=B*B                                                            FFM05170
      B3=B*B2                                                           FFM05180
      CALL LININT (RM,RM2,C,C2,X1(L),X2(L),F,P)                         FFM05190
      P30=P30+DI*(A*P(4)+B*P(3))                                        FFM05200
      F30=F30+DI*(A*F(4)+B*F(3))                                        FFM05210
      P21=P21+DI*(A2*P(4)+2*A*B*P(3)+B2*P(2))/2.                        FFM05220
      F21=F21+DI*(A2*F(4)+2*A*B*F(3)+B2*F(2))/2.                        FFM05230
      P12=P12+DI*(A3*P(4)+3*A2*B*P(3)+3*A*B2*P(2)+B3*P(1))/3.           FFM05240
      F12=F12+DI*(A3*F(4)+3*A2*B*F(3)+3*A*B2*F(2)+B3*F(1))/3.           FFM05250
      P20=P20+DI*(A*P(3)+B*P(2))                                        FFM05260
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```
      F20=F20+DI*(A*F(3)+B*F(2))
      P11=P11+DI*(A2*P(3)+2*A*B*P(2)+B2*P(1))/2.
      F11=F11+DI*(A2*F(3)+2*A*B*F(2)+B2*F(1))/2.
      P10=P10+DI*(A*P(2)+B*P(1))
      F10=F10+DI*(A*F(2)+B*F(1))
   22 CONTINJE
   33 CONTINJE
      RETURN
      END
C
      SUBROUTINE LININT (RM,RM2,C,C2,XS,XF,F,P)
      REAL P(4)
      COMPLEX C,C2,Z1,Z2,W1,W2,CLN,F0,F(4),D1,D2,CM,CM2
      CM=CMPLX(RM,0.0)
      CM2=CMPLX(RM2,0.0)
      XSK=XS
      XFK=XF
      DO 11 K=1,4
      AK=FLOAT(K+1)
      XSK=XS*XSK
      XFK=XF*XFK
   11 P(K)=(XFK-XSK)/AK
      W1=C*XS+CM
      W2=C*XF+CM
      Z1=C*XS-CM
      Z2=C*XF-CM
      IF(RM.GT.0.1)GJ TO 22
      IF(CABS(Z1).GT.1.0E-5) GJ TO 13
      Z1=(1.0,0.0)
      W1=(1.0,0.0)
   13 IF(CABS(Z2).GT.1.0E-5) GJ TO 22
      Z2=(1.0,0.0)
      W2=(1.0,0.0)
   22 CONTINJE
      D1=CLN(Z2)-CLN(Z1)
      D2=CLN(W2)-CLN(W1)
      F0=0.5*(D1-D2)/C
      F(1)=0.5*(D1+D2)/C2
      F(2)=((XF-XS)+CM*F0)/C2
      F(3)=(P(1)+CM2*F(1))/C2
      F(4)=(P(2)+CM2*F(2))/C2
      RETURN
      END
      FUNCTIJN CSR(Z)
C
      COMPLEX CSR,Z
      R=CABS(Z)
      P=0.5*ATAN2(AIMAG(Z),REAL(Z))
```

```
      CSR=SQRT(R)*CMPLX(COS(P),SIN(P))                                  FFM05750
      RETURN                                                            FFM05760
      END                                                               FFM05770
C                                                                       FFM05780
      FUNCTION CLN(Z)                                                   FFM05790
      COMPLEX CLN,Z                                                     FFM05800
      ZR=REAL(Z)                                                        FFM05810
      ZI=AIMAG(Z)                                                       FFM05820
      ZABS=SQRT(ZR*ZR+ZI*ZI)                                            FFM05830
      R=ALOG(ZABS)                                                      FFM05840
      FI=-1.0*ARCOS(ZR/ZABS)                                            FFM05850
      CLN=CMPLX(R,FI)                                                   FFM05860
      RETURN                                                            FFM05870
      END                                                               FFM05880
C                                                                       FFM05890
C                                                                       FFM05900
C                                                                       FFM05910
      SUBROUTINE INTEQ (Z,CMAG,CFAZ)                                    FFM05920
C     HALLEN'S I.E. SOLN OF THIN-WIRE SCATTERING                        FFM05930
C     ASSUMING INC EO = 1 V/M                                           FFM05940
C     NWIRE = NO WIRE PTS WITH ENDS                                     FFM05950
C     FREQ  = MHZ                                                       FFM05960
C     ZLEN  = WIRE LENGTH (M)                                           FFM05970
C     RAD   = WIRE RADIUS (M)                                           FFM05980
C     ZBOT  = Z AT WIRE BOTTOM (M)                                      FFM05990
C     NINC  = NO INC FLDS                                               FFM06000
C     DALPHA= ARRAY OF INC ANGLES (DEG) WITH EZ=EO SIN(ALPHA)           FFM06010
      COMPLEX A(20,20),U(20),ZI(20),C,D,E                               FFM06020
      REAL DALPHA(1),Z(20),CMAG(20),CFAZ(20)                            FFM06030
  100 FORMAT(2I5)                                                       FFM06040
      READ (5,100) NWIRE,NINC                                           FFM06050
      READ(5,101) FREQ,ZLEN,RAD,ZBOT                                    FFM06060
      READ(5,101) (DALPHA(I),I=1,NINC)                                  FFM06070
  101 FORMAT(5F10.3)                                                    FFM06080
      PI=3.1415927                                                      FFM06090
      DTR=PI/180.                                                       FFM06100
      RTD=180./PI                                                       FFM06110
      DZ=ZLEN/(NWIRE-1.)                                                FFM06120
      WL=300./FREQ                                                      FFM06130
      WN=2.*PI/WL                                                       FFM06140
      WRITE(6,102) NWIRE,NINC,FREQ,ZLEN,RAD,ZBOT,WL                     FFM06150
  102 FORMAT('1NWIRE,NINC,FREQ,ZLEN,RAD,ZBOT,WL:',/' ',2I6,5F9.3)       FFM06160
      CALL TWMAT(NWIRE,RAD,ZBOT,ZLEN,WL,A)                              FFM06170
C     INC FLD LOOP                                                      FFM06180
      DO 44 I=1,NINC                                                    FFM06190
      WRITE(6,103) DALPHA(I)                                            FFM06200
  103 FORMAT('0INC ALPHA (DEG):',F10.3,/)                               FFM06210
C     CHANGE ALPHA TO CORRESPOND TO MAIN PROGRAM                        FFM06220
```

```
      DALPHA(I) = 180.0-DALPHA(I)                                       FFM06230
      ALPHA=DTR*DALPHA(I)                                               FFM06240
      CA=COS(ALPHA)                                                     FFM06250
      SA=SIN(ALPHA)                                                     FFM06260
      DO 11 N=1,NWIRE                                                   FFM06270
      Z(N)=ZBOT+(N-1.)*DZ                                               FFM06280
      ERE=COS(WN*CA*Z(N))                                               FFM06290
      EIM=SIN(WN*CA*Z(N))                                               FFM06300
   11 U(N)=CMPLX(-EIM,ERE)/(377.*WN*SA)                                 FFM06310
      DO 22 N=1,NWIRE                                                   FFM06320
      ZI(N)=(0.,0.)                                                     FFM06330
      DO 22 M=1,NWIRE                                                   FFM06340
   22 ZI(N)=ZI(N)+A(N,M)*U(M)                                           FFM06350
      WRITE(6,104) ZI(1),ZI(NWIRE)                                      FFM06360
  104 FORMAT('OC AND D:',4E14.6,//'ON,Z,RE(I),IM(I),MAG(I),DEG(I):'/)   FFM06370
      ZI(1)=(0.,0.)                                                     FFM06380
      ZI(NWIRE)=(0.,0.)                                                 FFM06390
      DO 33 N=1,NWIRE                                                   FFM06400
      CMAG(N)=CABS(ZI(N))                                               FFM06410
      CFAZ(N)=0.0                                                       FFM06420
      IF(CMAG(N).LT.1.E-15) GO TO 32                                    FFM06430
      CFAZ(N)=RTD*ATAN2(AIMAG(ZI(N)),REAL(ZI(N)))                       FFM06440
   32 CONTINUE                                                          FFM06450
      WRITE(6,105) N,Z(N),ZI(N),CMAG(N),CFAZ(N)                         FFM06460
  105 FORMAT(' ',I7,F10.3,3E14.6,F10.3)                                 FFM06470
   33 CONTINUE                                                          FFM06480
   44 CONTINUE                                                          FFM06490
      RETURN                                                            FFM06500
      DEBUG SUBCHK                                                      FFM06510
      END                                                               FFM06520
C                                                                       FFM06530
      SUBROUTINE TWMAT(NX,RA,Z1,XL,XWAVE,TW)                            FFM06540
      COMPLEX TW(20,20),G(20),DET                                       FFM06550
      REAL X(20)                                                        FFM06560
      NMX = 20                                                          FFM06570
      N1=NX-1                                                           FFM06580
C     GENERATE N+2 NODES ON WIRE WITH Z1 AS FIRST NODE                  FFM06590
C     AND INCREMENTS OF 1/(N+1)                                         FFM06600
      DO 1 I=1,NX                                                       FFM06610
    1 X(I)=Z1+(I-1)*XL/(NX-1)                                           FFM06620
      P=XL/(NX-1)/3.                                                    FFM06630
      Q=P/2.                                                            FFM06640
      XKO=2*3.141593/XWAVE                                              FFM06650
C     LOAD THE C,D COEFFICIENT TERMS                                    FFM06660
      DO 2 I=1,NX                                                       FFM06670
      THETA=XKO*X(I)                                                    FFM06680
      TW(I,1)=SIN(THETA)                                                FFM06690
      TW(I,NX)=COS(THETA)                                               FFM06700
```

```
  2     CONTINUE                                                          FFM06710
C       LOAD REST OF COEFFICIENTS                                         FFM06720
        DO 3 I=1,NX                                                       FFM06730
        CALL GKRNL(G,X,XKO,RA,NX,I)                                       FFM06740
        DO 4 I1=2,N1                                                      FFM06750
        TW(I,I1)=Q*G(I1-1)+2*P*G(I1)+Q*G(I1+1)                            FFM06760
  4     CONTINUE                                                          FFM06770
  3     CONTINUE                                                          FFM06780
C       INVERT THE IMP MATRIX                                             FFM06790
        INORM=1                                                           FFM06800
        CALL CSMINV(TW,NMX,NX,DET,COND,INORM)                             FFM06810
        DMAG=CABS(DET)                                                    FFM06820
        WRITE(6,110) DMAG,COND                                            FFM06830
  110   FORMAT('OZWMAT DET AND COND:',2E11.3,/)                           FFM06840
        RETURN                                                            FFM06850
        DEBUG SUBCHK                                                      FFM06860
        END                                                               FFM06870
C                                                                         FFM06880
        SUBROUTINE GKRNL(G,X,XKO,RA,NX,M)                                 FFM06890
        REAL X(NX)                                                        FFM06900
        COMPLEX G(NX)                                                     FFM06910
        PI=3.1415927                                                      FFM06920
        XM=X(M)                                                           FFM06930
        DO 11 N=1,NX                                                      FFM06940
        XN=X(N)                                                           FFM06950
        R=SQRT((XN-XM)*(XN-XM)+RA*RA)                                     FFM06960
  1     FORMAT(10X,E12.4)                                                 FFM06970
        C=COS(XKO*R)                                                      FFM06980
        S=-1.*SIN(XKO*R)                                                  FFM06990
        G(N)=CMPLX(C,S)/(4.0*PI*R)                                        FFM07000
  11    CONTINUE                                                          FFM07010
        RETURN                                                            FFM07020
        END                                                               FFM07030
C                                                                         FFM07040
C                                                                         FFM07050
C                                                                         FFM07060
        SUBROUTINE HIGEN (RM,DZ,NW,IEX,HB,NB,ZL)                          FFM07070
        COMPLEX HB(NB)                                                    FFM07080
        REAL IEX(NW)                                                      FFM07090
        DO 10 I = 1,NW                                                    FFM07100
        ZNW = DZ*(2.0*I-NB)/2.0                                           FFM07110
        IEX(I) = SIN(ZL+ZNW)                                              FFM07120
        IF(ZNW.GT.0.0) IEX(I) = SIN(ZL-ZNW)                               FFM07130
  10    CONTINUE                                                          FFM07140
        DO 20 I = 1,NB                                                    FFM07150
        ZNB = DZ*(2.0*I-NB)/2.0                                           FFM07160
        RK = SQRT(ZNB*ZNB+RM*RM)                                          FFM07170
        R1K = SQRT((ZNB-ZL)**2+RM*RM)                                     FFM07180
```

```
      R2K = SQRT((ZNB+ZL)**2+RM*RM)                                     FFM07190
      APART = SIN(R1K)+SIN(R2K)-2.*SIN(RK)*COS(ZL)                      FFM07200
      BPART = COS(R1K)+COS(R2K)-2.*COS(RK)*COS(ZL)                      FFM07210
      HB(I) = (30.0*6.283185/RM)*CMPLX(APART,BPART)                     FFM07220
   20 CONTINUE                                                          FFM07230
      RETURN                                                            FFM07240
      END                                                               FFM07250
C     /*                                                                FFM07260
//GO.PLOTPARM  DD  *                                                    FFM07270
 &PLOT  SCALE = 0.7,YSTART = 2.0,STRIPO=2.0 $END                        FFM07280
C     /*                                                                FFM07290
//GO.SYSIN  DD  *                                                       FFM07300
   20    1                                                              FFM07310
 300.0     0.5       0.02      -.25                                     FFM07320
  90.0                                                                  FFM07330
CURRENT MAGNITUDE :FEM/T=SOLID :INT.EQ=X                                FFM07340
ZL=.5:AL=0.02:BL=0.001:ALFA=90:NW=40                                    FFM07350
CURRENT PHASE :FEM/T=SOLID :INT.EQ=PLUS                                 FFM07360
ZL=.5:AL=0.02:BL=0.001:ALFA=90:NW=40                                    FFM07370
WIRE CURRENT MAGNITUDE:AL=.02:BL=.001                                   FFM07380
SOLID = EXACT : DIAMONDS = FEM : NW=40                                  FFM07390
WIRE CURRENT PHASE : ZL=.5 :BL=0.001                                    FFM07400
CROSSES = EXACT : TRIANGLES = FEM : NW=40                               FFM07410
BOUNDARY FIELD MAG. : ZL=.5 : DR=DZ/9.8                                 FFM07420
SOLID=EXACT : DIAMONDS=T : AL=1/100                                     FFM07430
BOUNDARY FIELD PHASE : ZL=.5 : DR=DZ/9.8                                FFM07440
SOLID=EXACT : TRIANGLES=T : AL=1/100                                    FFM07450
```

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